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THERMAL POSTBUCKLING OF HEATED UNIFORM COLUMNS CONSIDERING GREEN NONLINEARITY: A NOVEL LINEAR FINITE ELEMENT FORMULATION

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Abstract

Thermal postbuckling behavior of uniform, heated columns is presented, by using a novel linear Finite Element (FE) formulation. In this investigation, the general Green axial nonlinear strain- displacement relation is used, instead of the popularly used simpler von-Karman nonlinearity, which is a subset of Green nonlinearity. In the earlier complex nonlinear FE formulations, time consuming iterative or step-by-step methods are used to obtain the solution for thermal postbuckling. In the novel FE formulation, normally used FEs , for performing linear buckling analysis, is proposed to obtain thermal postbuckling loads. The nodal degrees of freedom are deflection and its first derivative with respect to the axial coordinate. The geometric nonlinearity is incorporated through the tensile loads induced, with axially re-strained ends of the column, due to large deflections. The effectiveness of the novel FE formulation is demonstrated, from the numerical results obtained, in terms of the ratio of thermal postbuckling to buckling loads, for specified reference deflection and slenderness ratios, of the columns with different boundary conditions. The numerical results reveal some interesting phenomena of thermal postbuckling behavior of columns.

Keywords*: Thermal Loads; Thermal Buckling; Thermal Postbuckling; Novel Finite Element Formulation; Linear Eigenvalue Problem*

Introduction

The importance of predicting the thermal postbuckling behavior of the structural members has been recognized, where these can withstand additional temperature beyond the buckling temperature, when deflections are large. Several structural systems, which are subjected to high temperatures in their service conditions, are assemblages of these structural members, like the columns, plates and shells. The temperature rise ΔT, which is above the stress free temperature T_{sf}, produces a mechanical equivalent of compressive loads (or thermal loads) that cause thermal buckling, and subsequently thermal postbuckling with large deflections. A proper understanding of this phenomenon of thermal postbuckling and its prediction is

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necessary, to achieve competitive and usable structural systems, in many fields of engineering.

A novel linear Finite Element (FE) formulation, is proposed in this paper, to investigate thermal postbuckling behavior of heated structural members. In this study, for a better and easy understanding of the novel FE formulation, the simplest compressive load carrying structural member (column), with respect to formulation, is chosen as a demonstration problem, to predict its thermal postbuckling behavior. However, the proposed novel FE formulation can be applied for other structural members, where the thermal buckling loads can be evaluated, by using the linear eigenvalue extraction algorithms.

The complex geometric nonlinear thermal postbuckling behavior of columns and other structural members, with some complicating effects, has been studied by many researchers, either theoretically (continuum mechanics) or numerically (FE) method) [1-7]. In these studies, the popular and simplified von-Karman geometric nonlinear axial strain-displacement relation(s), which is applicable for moderately large deflections, which are of the order of the characteristic dimension, namely, radius of gyration of the cross-section r of the column or thickness t for other structural members. In the von-Karman nonlinear theory, the nonlinearity of deflection only is considered in thermal postbuckling analysis, where the nonlinearity involved in axial displacement is neglected. This assumption, though simplifies the formulation, but puts a constraint on the magnitude of the deflections, which have to be moderately large.

The proposed novel FE formulation is based on Green nonlinear strain-displacement relation [8] that considers both the nonlinearities involved in the axial displacement and deflection, from which thermal postbuckling phenomenon can be analyzed, without any restriction on the magnitude of deflection. It is rather difficult, if not impossible, to use the continuum mechanics formulations, to predict thermal postbuckling behavior of the columns. Alternatively, use of the earlier matrix structural analysis, which is called later as the FE method [9, 10], provides a reliable and accurate solution for thermal postbuckling analysis of columns, by using von-Karman nonlinearity, which is a subset of Green nonlinearity.

In this investigation, as has been already mentioned, a novel linear FE formulation, is proposed to predict thermal postbuckling behavior of uniform columns, by using the linear element stiffness and geometric stiffness matrices. The proposed FE formulation of columns, can be used, with some minimal changes, to other structural members like circular and rectangular plates, as the corresponding linear element stiffness and geometric stiffness matrices are readily available [9, 10]. The compressive mechanical equivalent of thermal load P_t is evaluated by following the procedure given in Ref. [11]. If the two ends of column are restrained to move axially, and is subjected to a uniform temperature rise (ΔT), from stress free temperature (T_{sf}) , the magnitude of this load P_t is given by '$E\,A\,\alpha\,\Delta T$', where E is Young's modulus, and a is coefficient of linear thermal expansion of the material of the column and A is its area of cross-section. Since, the present work considers thermal postbuckling behavior, the subscript 't' used to represent 'thermal', is omitted in all symbols used in this paper, from now onwards. For example, the thermal buckling and postbuckling loads are denoted by P_b and P_{pb} instead of P_{b_t} and P_{pb_t} . Thermal postbuckling load carrying capacity is generally represented, as the ratio $\frac{P_{pb}}{P_b}$, for a specified central lateral deflection ratio $\frac{b}{r}$, where b is the reference deflection, which is generally taken at the mid- length of column $(x = \frac{L}{2})$, where x is axial coordinate and L is length of column.

In the following sections, the proposed novel FE formulation is presented, where the usual linear element elastic stiffness and geometric stiffness matrices, which are of order 4 by 4 [9,10], instead of using higher order (8 by 8) nonlinear element stiffness, and geometric stiffness matrices [12]. Another important feature of the proposed FE method is that the iterative or step-by-step procedure is required in earlier FE formulations [13, 14], is not required to obtain the solution. As a consequence, the proposed FE formulation reduces the computational time by orders of magnitude, since a linear eigenvalue problem only has to be solved. The boundary conditions on deflection, taken at two ends of the column, are hinged-hinged (***h-h***), clamped-hinged (***c-h***) and clamped-clamped (***c-c***).

Green Nonlinear Axial Strain-Displacement Relation

In the novel FE formulation, Green nonlinear axial strain-displacement relation, for one-dimensional problems, like columns, are given [8] by

$$\varepsilon_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{1}$$

where ε_x is axial strain, u is axial displacement and w is deflection and x is axial coordinate. It is to be noted that the nonlinear strain-displacement relation given in Eq.(1), is general and does not have any limitation on magnitude of deflection.

Earlier researchers, to quote a few, presented their formulations [1-7], by neglecting the nonlinear term corresponding to $u\left[=\left(\frac{du}{dx}\right)^2\right]$, when compared to nonlinear term in $w\left[=\left(\frac{dw}{dx}\right)^2\right]$, based on their magnitudes. This imposes a restriction that the deflections are moderately large, while studying thermal postbuckling of columns, and implies that the axial displacement u is order of magnitude smaller than deflection w depending on the value of slenderness ratio of column, $SR\ (=\frac{L}{r})$.With this restriction, Eq.(1), is rewritten as

$$\varepsilon_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{2}$$

Equation(2)ispopularlyknownasvon-Karmannonlinear axial strain-displacement relation. This equation simplifies thermal postbuckling formulation of the heated columns, when compared to the same, based on Green nonlinear axial strain-displacement relation.

It is to be noted that the use of von-Karman nonlinearity is adequate for many engineering applications, and consideration of Green nonlinearity is general, which is applicable, even when deflections are large, and reveals some interesting phenomena related to thermal postbuckling of columns. The linear novel FE formulation proposed here is an effort to show how the general Green nonlinear formulation can be taken care of easily, to predict thermal postbuckling behavior of columns.

Evaluation of Induced Tensile Loads T_u and T_w

The necessary input required to take care of Green geometric nonlinearity, in the proposed FE formulation, is constant tensile load T which is algebraic sum of two parts that are given by constant tensile loads T_u and T_w, which are induced in the column, due to large deflections, when two ends of column are restrained to move axially. Evaluation of these tensile loads T_u and T_w is presented here by following the procedure given in Ref.[15].

Consider a column, wherein one end of column is free to move axially, and having a constraint on axial immovability at the other end. If an axial tensile load T_a, where subscript 'a' corresponds to either u or w, applied at free end. By following the procedure, given in Ref.[11], the outward axial displacement $u_{o,a}$, at this free end of column is given, by

$$u_{o,a} = \frac{T_a L}{EA} \tag{3}$$

If the column undergoes large deflections, then the inward axial displacement $u_{i,a}$, at free end of column [15], is given by

$$u_{i,a} = \frac{1}{2}\int_o^l (u')^2\, dx \tag{4}$$

By following Dym [8], the relation, with a small approximation, between the derivatives u and w is given, by

$$u' \cong -\frac{1}{2}(w')^2 \tag{5}$$

where $(\ )'$ represents first derivative with respect to x, and Eq.(4) can be written, as

$$u_{i,a} = \frac{1}{8}\int_o^l (w')^4\, dx \tag{6}$$

To satisfy the axially immovable condition at both ends of column, magnitudes of these two axial displacements $u_{o,a}$ and $u_{i,a}$ have to be equal, which gives the magnitude of T_u, as

$$T_u = \frac{EA}{8L}\int_o^l (w')^4\, dx \tag{7}$$

Integral in Eq.(7) is evaluated numerically, by using w' Degree of Freedom (DOF) at the nodes of discretized column, as explained in novel FE formulation, which is presented in the next section.

The aforementioned procedure is followed, to evaluate constant tensile load T_w induced in the column, due to nonlinearity in deflection w that exists in Green nonlinear axial strain-displacement relation, as

$$T_w = \frac{E\,A}{2\,L} \int_o^L (w\,')^2 \; dx \qquad (8)$$

Integral in Eq.(8) is also evaluated as in the case of T_w.

Incidentally, the constant tensile load T_W given in Eq.(8), is due to von-Karman nonlinearity. The constant tensile loads induced in the column, due to large deflections, are written in non-dimensional form as $\boldsymbol{\lambda_{t_u}} = \left(\frac{T_u L^2}{E\,I}\right)$ and $\boldsymbol{\lambda_{t_w}} = \left(\frac{T_W L^2}{E\,I}\right)$. It is to be noted that all non-dimensional quantities, parameter and equations are written in **bold type**, from now onwards. Though it is not explicitly seen in Eqs.(7) and (8), $\boldsymbol{\lambda_{t_u}}$ parameter is directly proportional to mid-length deflection ratio $\left(\frac{b}{r}\right)^4$ and inversely proportional to $\boldsymbol{SR}^2$, where b is mid-length deflection and ***SR*** is the slenderness ratio $\frac{L}{r}$ column, and $\boldsymbol{\lambda_{t_w}}$ is directly proportional to $\left(\frac{b}{r}\right)^2$, and is independent of ***SR***.

The total constant axial tensile load parameter $\boldsymbol{\lambda_t} = \left(\frac{TL^2}{E\,I}\right)$, which is induced in the column, by considering general Green nonlinear strain-displacement relation, is obtained as, $\boldsymbol{\lambda_t} = \boldsymbol{\lambda_{t_u}} + \boldsymbol{\lambda_{t_w}}$, which is a measure of the total geometric nonlinearity of column that is used to evaluate the expression for the ratio of thermal postbuckling to buckling load parameter $\frac{\lambda_{pb}}{\lambda_b}$, for a specified value of mid-length lateral deflection ratio $\frac{b}{r}$ and ***SR***. If $\frac{b}{r} \to 0$, the value of the total tensile load parameter $\boldsymbol{\lambda_t} \to 0$, and as a result, the geometric nonlinearity involved in thermal postbuckling problem does not exist. In such a situation, thermal postbuckling load, P_{pb}, or in non-dimensional form $\boldsymbol{\lambda_{pb}} = \left(\frac{P_{pb} L^2}{E\,I}\right)$ of the column, becomes thermal buckling load P_b, or in non-dimensional form $\boldsymbol{\lambda_b} = \left(\frac{P_b L^2}{E\,I}\right)$. By following the logical steps of Ref.[13], the ratio $\frac{\lambda_{pb}}{\lambda_b}$ is written in a simple form, as

$$\frac{\lambda_{pb}}{\lambda_b} = 1 + \frac{\lambda_t}{\lambda_b} = 1 + \frac{\lambda_{t_u} + \lambda_{t_w}}{\lambda_b} \qquad (9)$$

Novel FE Formulation

The main motivation of this study is to propose a linear novel FE methodology, to predict thermal postbuckling behavior of heated uniform columns, when the temperature rise ΔT defined earlier is constant along the length and across the cross-section of column. ΔT produces a mechanical equivalent of constant compressive thermal load, if the two ends of column are restrained to move axially. In the novel FE formulation, standard linear 4 by 4 element stiffness [k], and the same order element geometric stiffness [g] matrices are used. Details of the derivation of these commonly used element matrices are not given here, as these are available in Refs.[8, 9]. One-dimensional, straight beam FE_S, which are used for the final linear eigenvalue analysis, have two nodes at the ends and two DOF, w and $\frac{dw}{dx}$ ($= w\,'$) at the nodes, which are the deflection w and its first derivative with respect to the axial coordinate x of column. The important feature of this FE formulation, when compared to other formulations, is that the DOF corresponding to axial displacement u and its first derivative $\frac{du}{dx}$ are not necessary, even for the nonlinear problem, like thermal postbuckling analysis. The effect of geometric nonlinearity is introduced here from the constant total axial tensile load parameter $\boldsymbol{\lambda_t}$. Equal length beam FE_S are used to discretize the column. Global elastic stiffness [K] and geometric stiffness [G] matrices, are obtained by following standard procedure [9,10]. The order of these global matrices before applying boundary conditions, depends on the number of nodes (n), where n is 2 (***NE*** +1), and ***NE*** is number of equal length FE_S with which the column is discretized. The accuracy of evaluation of thermal postbuckling load, depends on the order of the global matrices after applying boundary conditions.

In the novel FE formulation, the value of induced tensile load parameter $\boldsymbol{\lambda_t}$ in the column, is treated as a constant initial tensile load acting along its length, and following non-dimensional matrix equilibrium equation is obtained [16], as

$$[\boldsymbol{K}]\{\boldsymbol{\delta}\} + \boldsymbol{\lambda_t}[\boldsymbol{G}]\{\boldsymbol{\delta}\} - \boldsymbol{\lambda_{pb}}[\boldsymbol{G}]\{\boldsymbol{\delta}\} = \{\boldsymbol{0}\} \qquad (10)$$

where $\{\boldsymbol{\delta}\}$ is non-dimensional eigenvector and $\{\mathbf{0}\}$ is a null vector, which is rewritten in the following form, as

$$[K]\{\delta\} - \left(\lambda_{pb} - \lambda_t\right)[G]\{\delta\} = \{0\} \qquad (11)$$

Equation (11) represents a linear eigenvalue problem, and after applying proper boundary conditions, is solved by using any standard algorithm that extracts eigenvalues (buckling load parameters) and eigenvectors (buckling mode shapes). The lowest eigenvalue is thermal buckling load parameter $\boldsymbol{\lambda_b}$, and the corresponding linear eigenvector $\{\delta\}$ contains relative values of $\boldsymbol{w}$ and w', which are non-dimensionalized with respect to the mid-length deflection ratio $\frac{b}{r}$, so that the corresponding normalized eigenvector contains the specified value of mid-length deflection $\frac{b}{r}$. From this eigenvector the value of $\boldsymbol{\lambda_t} \; (= \lambda_{t_u} + \lambda_{t_w})$ can be evaluated, for a specified ***SR***.

Equation (11) is solved, after applying boundary conditions of the column, by following standard procedure used in FE analysis [9,10], to obtain the value of $\boldsymbol{\lambda_b}$, which is equal to $(\lambda_{pb} - \lambda_t)$, and the corresponding normalized eigenvector. This procedure is repeated for different values of the specified number of mid-length lateral deflection ratios $\frac{b}{r}$ and slenderness ratios ***SR***, to obtain thermal postbuckling load parameter $\boldsymbol{\lambda_{pb}}$ of the heated column for these ratios. As mentioned earlier, as a degenerate case, the linear buckling load parameter $\boldsymbol{\lambda_b}$ of the column can be obtained by treating the constant tensile load parameter $\lambda_t = 0$, which means that geometric nonlinearity is not considered in the FE analysis. The number of Finite Elements (NE) required, which varies with respect to boundary conditions of column, mid-length deflection ratio $\frac{b}{r}$ and ***SR***, to achieve convergence of $\boldsymbol{\lambda_b}$ or $\boldsymbol{\lambda_t}$ that contain λ_{t_u} and λ_{t_w}, to a specified accuracy of the numerical results. For typical unsymmetrical boundary condition, namely, the clamped-hinged (***c-h***) column, all parameters and ratios have converged, with 64 elements, to achieve the specified accuracy of five significant figures. The complexity of the boundary conditions decrease from the clamped-clamped (***c-c***) to hinged-hinged (***h-h***) columns, which require 32 and 16 elements for convergence, to achieve the same specified accuracy. An important advantage of the proposed novel FE formulation is that convergence study with respect to the number of FE_S is the only criterion, to obtain the values of $\boldsymbol{\lambda_t}$, which contains λ_{t_u} and λ_{t_w}, $\boldsymbol{\lambda_b}$ and $\boldsymbol{\lambda_{pb}}$ and subsequently the ratio of thermal postbuckling to thermal buckling loads $\frac{\lambda_{pb}}{\lambda_b}$ accurately, for the values of $\frac{b}{r}$ and ***SR*** considered in this study.

Numerical Results and Discussion

The proposed linear novel FE formulation, is used to obtain thermal postbuckling behavior, considering Green nonlinearity in terms of $\frac{\lambda_{pb}}{\lambda_b}$ of the heated uniform and isotropic columns, with different boundary conditions on deflection, for a specified central deflection ratio $\frac{b}{r}$ and slenderness ratio $\boldsymbol{SR}(=\frac{L}{r})$ This ratio is $\frac{\lambda_{pb}}{\lambda_b}$ evaluated from the constant tensile loads λ_{t_u} and λ_{t_w}. An important requirement of the FE formulation is to study the convergence of $\boldsymbol{\lambda_b}$ and constant total tensile load $\boldsymbol{\lambda_t}$, with the number of equal length FE_S (NE) with which the column is discretized, where $\boldsymbol{\lambda_t}$ is the algebraic sum of two constant tensile loads λ_{t_u} and λ_{t_w} induced in the column undergoing large deflections, due to the nonlinearities existing in the axial displacement and deflection, in the general Green strain-displacement relation. The boundary conditions of column, in terms of the end deflection *w* and its first derivative *w* ′, considered in this study are the ***h-h***, ***c-h*** and ***c-c***, where ***c*** (*w* = 0 and *w* ′ = 0) and ***h*** (*w* = 0) represent the clamped and hinged boundary conditions.

Table-1 presents, the convergence of the thermal buckling load parameter $\boldsymbol{\lambda_b}$ for the ***h-h***, ***c-h*** and ***c-c*** columns. For the ***h-h*** column, the maximum value of $\frac{b}{r}$ is taken as 10.0, and for the ***c-h*** and ***c-c*** columns, the maximum value of $\frac{b}{r}$ is taken as 16.0. A good convergence is achieved for the three boundary conditions, namely, for ***h-h***, ***c-h*** and ***c-c*** columns with 16, 64 and 32 element discretization, respectively. It is seen from this Table that the converged values of $\boldsymbol{\lambda_b}$ match very well with those given by Timoshenko and Gere [2].

The convergence study of constant tensile loads λ_{t_u} and λ_{t_w} are presented in Table-2 for all three boundary conditions of column. It is already mentioned that λ_{t_u} is directly proportional to $\left(\frac{b}{r}\right)^4$ and inversely proportional to SR^2, and λ_{t_w} is directly proportional to $\left(\frac{b}{r}\right)^2$ and is independent of ***SR***. For the ***h-h*** column the value of λ_{t_u} and λ_{t_w} are evaluated for the maximum value of $\frac{b}{r} = 10.0$ and ***SR*** = 60.0 (middle value of the three values of ***SR*** considered). For the ***c-h*** and ***c-c*** columns, the values of λ_{t_u} and λ_{t_w} are evaluated at the maximum value of $\frac{b}{r} = 16.0$ and ***SR*** = 80.0 (again the middle value). The convergence of λ_{t_u} and λ_{t_w} is achieved, with the same NE as in the case of λ_b , and ensures convergence of $\frac{\lambda_{pb}}{\lambda_b}$, as this ratio is a direct function of λ_{t_u} and λ_{t_w} , for the three boundary conditions considered.

Table-1 : Convergence Study of Buckling Load Parameter λ_b			
NE	**h-h** Column	**c-h** Column	**c-c** Column
2	9.9438	20.7088	40.0000
4	9.8747	20.2322	39.7754
8	9.8699	20.1935	39.4986
16	9.8696	20.1909	39.4797
32	---	20.1907	39.4784
Ref.[2]	9.8696	20.19	39.4784

It is to be noted that the relatively poor convergence of λ_b for ***c-h*** column is due to the unsymmetric configuration of the column, when compared to the ***h-h*** and ***c-c*** columns. The similar phenomenon for λ_{t_u} and λ_{t_w} is that the reference (maximum) deflection is taken at the mid-length of the columns. While this is acceptable for the ***h-h*** and ***c-c*** columns, for the ***c-h*** column the mid-length deflection is not the maximum deflection, but the maximum deflection occurs at a point nearer to the hinged end of the column, and hence the convergence of λ_{t_u} and λ_{t_w} is poor for the ***c-h*** column.

Table-3 presents the results of the present study with those given in Ref.[17]. From this Table, it can be observed that different parameters β, and *m* are used to present the results for $\frac{\lambda_{pb}}{\lambda_b}$ in Ref.[17] separately for the ***h-h*** and ***c-c*** columns. For a better understanding of the results of Ref.[17], it is necessary to obtain $\frac{b}{r}$ corresponding to β and *m*. The corresponding expressions that convert β and *m* are given by

Table-2 : Convergence Study of Tensile Load Parameters λ_{t_u} or λ_{t_w}						
NE	**h-h** Column		**c-h** Column		**c-c** Column	
	$\lambda_{t_u}$$	λ_{t_w}+	λ_{t_u}#	λ_{t_w}@	λ_{t_u}#	λ_{t_w}@
2	12.684	116.48	---	---	---	838.96
4	12.684	246.73	71.823	775.68	---	838.96
8	12.684	246.74	66.329	666.39	48.369	631.68
16	12.684	246.74	62.016	666.14	46.758	631.65
32	---	---	61.153	666.14	46.758	631.65
64	---	---	61.144	---	---	---
$ SR = 60.0 and $\frac{b}{r} = 10.0$			+ Independent of SR and $\frac{b}{r} = 10.0$			
# SR = 80.0 and $\frac{b}{r} = 16.0$			@ Independent of SR and $\frac{b}{r} = 16.0$			

Table-3 : Comparison of Present Results with Ref.[17]

h-h Column, SR = 120 (NE = 16)						Absolute Value of % difference	**c-c** Column, SR = 160 (NE = 32)						Absolute Value of % difference
β	$\frac{b}{r}$	λ_{t_u}	λ_{t_w}	$\frac{\lambda_{pb}}{\lambda_b}$ Present	$\frac{\lambda_{pb}}{\lambda_b}$ Ref.[17]		m	$\frac{b}{r}$	λ_{t_u}	λ_{t_w}	$\frac{\lambda_{pb}}{\lambda_b}$ Present	$\frac{\lambda_{pb}}{\lambda_b}$ Ref.[17]	
2	1.3328	0.0010	4.3829	1.4441	1.4438	0.0270	0.2	1.6211	0.0012	6.4842	1.1642	1.1638	0.0411
4	2.6623	0.0159	17.4885	2.7735	2.7764	0.1017	0.4	3.2422	0.0197	25.9369	1.6574	1.6563	0.0718
6	3.9854	0.0799	39.1907	4.9789	5.0019	0.4586	0.6	4.8634	0.0997	58.3606	2.4808	2.4816	0.0315
8	5.2990	0.2500	69.2831	8.0451	8.1265	1.0006	0.8	6.4855	0.3155	103.7831	3.6368	3.6464	0.2619
10	6.6002	0.6017	107.4865	11.9516	12.1590	1.7055	1.0	8.1056	0.7699	162.1101	5.1257	5.1604	0.6705
12	7.8860	1.2263	153.4452	16.6714	17.1130	2.5799	1.2	9.7268	1.5965	233.4424	6.9535	7.0365	1.1781
14	9.1540	2.2264	206.7576	22.1745	23.0010	3.5932	1.4	11.3470	2.9568	317.6888	9.1220	9.2914	1.8228
16	10.4016	3.7117	266.9562	28.4243	29.8420	4.7503	1.6	12.9690	5.0458	415.0044	11.6399	11.9460	2.5617
-	-	-	-	-	-		1.8	14.5900	8.0820	525.2310	14.5089	15.0240	3.4282
-	-	-	-	-	-		2.0	16.2110	12.3179	648.4244	17.7367	18.5600	4.4356

$$\frac{b}{r} = \frac{\beta}{180}\,(\boldsymbol{SR}) \qquad (12)$$

and

$$\frac{b}{r} = \frac{m}{2\,\pi^2}\,(\boldsymbol{SR}) \qquad (13)$$

The comparison of $\frac{\lambda_{pb}}{\lambda_b}$ obtained from the present study, compare well with those given in Ref. [17], after changing the parameters β and *m* to $\frac{b}{r}$. It is to be noted here that the $\frac{b}{r}$ value after modification from the parameters of Ref.[17] are not simple as given in other Tables, but contain decimal numbers, which can be seen from the results presented in Table-3.

The values of the parameters λ_{t_u}, λ_{t_w} and the ratio of thermal postbuckling to buckling loads $\frac{\lambda_{pb}}{\lambda_b}$ of the ***h-h*** columns for several values of $\frac{b}{r}$ varying from 2.0 to 10.0 in steps of 1.0 and for the values of ***SR*** equals to 30.0, 60.0 and 120.0 are presented in Table-4. In Tables-5 and 6 similar values for the ***c-h*** and ***c-c*** columns are presented for several values of $\frac{b}{r}$ varying from 2.0 to 16.0 in steps of 2.0, and for the values of ***SR*** equals to 40.0, 80.0 and 160.0. As has been mentioned earlier, the values of λ_{t_u} and λ_{t_w} increase with $\frac{b}{r}$, whereas the value of λ_{t_u} decreases with increasing ***SR***, and the value of λ_{t_w} is independent of ***SR***. This indicates that the constant tensile load parameter, which arises from the consideration of the nonlinearity in the axial displacement due to the nonlinearity in axial displacement is smaller for higher ***SR***, when compared to lower ***SR***. However, the ratio $\frac{\lambda_{pb}}{\lambda_b}$ increases with the increasing ratio $\frac{b}{r}$, and increasing ***SR***. The effect of higher ***SR*** from 120.0, makes the value of the parameter λ_{t_u} very small, when compared to the value of the parameter λ_{t_w}, which is significant. This reveals an important phenomenon that for higher values of ***SR*** the von-Karman nonlinearity (nonlinearity in *w*) is sufficient to predict the thermal postbuckling behavior, and consideration of the general Green nonlinearity is not necessary. On the other hand, when the value of ***SR*** is smaller, in which case the contribution of both the nonlinearities corresponding to the axial displacement and lateral deflection are significant. In such a situation, the effect of transverse shear

Table-4 : Values of λ_{t_u}, λ_{t_w} and $\frac{\lambda_{pb}}{\lambda_b}$ of h-h Column (NE = 16)

$\frac{b}{r}$	**SR** = 30.0			**SR** = 60.0			**SR** = 120.0		
	λ_{t_u}	λ_{t_w}	$\frac{\lambda_{pb}}{\lambda_b}$	λ_{t_u}	λ_{t_w}	$\frac{\lambda_{pb}}{\lambda_b}$	λ_{t_u}	λ_{t_w}	$\frac{\lambda_{pb}}{\lambda_b}$
2.0	0.081176	9.8696	2.0082	0.020294	9.8696	2.0021	0.0050735	9.8696	2.0005
3.0	0.41096	22.207	3.2916	0.10274	22.207	3.2609	0.025685	22.207	3.2526
4.0	1.29992	39.478	5.1316	0.32498	39.478	5.0329	0.081245	39.478	5.0082
5.0	3.17088	61.685	7.5712	0.79272	61.685	7.3303	0.19818	61.685	7.2701
6.0	6.5752	88.826	10.6661	1.6438	88.826	10.166	0.41095	88.826	10.042
7.0	12.1812	120.90	14.4839	3.0453	120.90	13.558	0.76130	120.90	13.327
8.0	20.7808	157.91	19.1051	5.1952	157.91	17.526	1.2985	157.91	17.131
9.0	33.3664	199.86	24.6307	8.3416	199.86	22.095	2.0854	199.86	21.462
10.0	50.592	246.74	31.1260	12.648	246.74	27.285	3.1710	246.74	26.321

Table-5 : Values of λ_{t_u}, λ_{t_w} and $\frac{\lambda_{pb}}{\lambda_b}$ of c-h Column (NE = 64)

$\frac{b}{r}$	**SR** = 40.0			**SR** = 80.0			**SR** = 160.0		
	λ_{t_u}	λ_{t_w}	$\frac{\lambda_{pb}}{\lambda_b}$	λ_{t_u}	λ_{t_w}	$\frac{\lambda_{pb}}{\lambda_b}$	λ_{t_u}	λ_{t_w}	$\frac{\lambda_{pb}}{\lambda_b}$
2.0	0.05972	10.408	1.5184	0.014930	10.408	1.5162	0.0035575	10.408	1.5157
4.0	0.95536	41.634	3.1093	0.23884	41.634	3.0739	0.059710	41.634	3.0650
6.0	4.8368	93.676	5.8791	1.2092	93.676	5.6994	0.30229	93.676	5.6545
8.0	15.286	166.54	10.0054	3.8215	166.54	9.4376	0.95538	166.54	9.2597
10.0	37.3192	260.21	15.7359	9.3298	260.21	14.350	2.3325	260.21	14.003
12.0	77.384	374.70	23.3907	19.346	374.70	20.516	4.8366	374.70	19.798
14.0	143.368	510.01	33.3603	35.842	510.01	28.035	8.9604	510.01	26.703
16.0	244.576	666.14	46.1057	61.144	666.14	37.021	15.286	666.14	34.750

deformation has to be considered [18]. Though the effect of transverse shear deformation is significant for smaller values of ***SR***, it affects only λ_b and consequently $\frac{\lambda_{pb}}{\lambda_b}$, and does not affect the two geometric nonlinear terms λ_{t_u} and λ_{t_w}.

Conclusions

The major conclusions, based on the present investigation, are briefly summarized below:

- In the novel FE formulation, thermal postbuckling problem is solved as a linear eigenvalue problem, and as such it is not necessary to use the tedious iterative or step-by-step method (with load increments).
- The convergence, which depends on the boundary conditions and use of reference deflection point of the column, of the thermal buckling load parameter λ_b, λ_{t_u}, λ_{t_w} and $\frac{\lambda_{pb}}{\lambda_b}$ for specified values of central deflection ratio $\frac{b}{r}$ and slenderness ratio ***SR*** is good.

Table-6 : Values of λ_{t_u}, λ_{t_w} and $\frac{\lambda_{pb}}{\lambda_b}$ of c-c Column (NE = 32)

$\frac{b}{r}$	**SR** = 40.0			**SR** = 80.0			**SR** = 160.0		
	λ_{t_u}	λ_{t_w}	$\frac{\lambda_{pb}}{\lambda_b}$	λ_{t_u}	λ_{t_w}	$\frac{\lambda_{pb}}{\lambda_b}$	λ_{t_u}	λ_{t_w}	$\frac{\lambda_{pb}}{\lambda_b}$
2.0	0.04566	9.8696	1.2511	0.011415	9.8696	1.2503	0.0028538	9.8696	1.2501
4.0	0.73056	39.478	2.0184	0.18264	39.478	2.0046	0.045605	39.478	2.0011
6.0	3.697	88.826	3.3436	0.92425	88.826	3.2734	0.23103	88.826	3.2558
8.0	11.6892	150.91	5.1186	2.9223	150.91	4.7486	0.73057	150.91	4.8411
10.0	28.538	246.74	7.9728	7.1345	246.74	7.4307	1.7836	246.74	7.2952
12.0	59.176	355.31	11.4990	14.794	355.31	10.375	3.6985	355.31	10.094
14.0	109.632	483.61	16.0270	27.408	483.61	13.944	6.8519	483.61	13.424
16.0	187.024	631.85	21.7423	46.756	631.85	18.189	11.689	631.85	17.301

- The values of λ_{t_u}, λ_{t_w} and $\frac{\lambda_{pb}}{\lambda_b}$ are presented for ***h-h*** column, when the deflection ratio $\left(\frac{b}{r}\right)$ is at the mid-length of the column, for the values of ***SR*** equal to 30.0, 60.0 and 120.0. As mentioned earlier, the values of $\frac{\lambda_{pb}}{\lambda_b}$ are more, for the lower value of *SR*. It is due to the fact that λ_{t_u} is inversely proportional to $(SR)^2$, and λ_{t_w} is independent of SR. For the ***c-h*** and ***c-c*** columns, the values of $\left(\frac{b}{r}\right)$ are at the mid-length of the column and SR equals to 40.0, 80.0 and 160.0. The conclusions of ***h-h*** column are valid for the variation of λ_{t_u}, λ_{t_w} and $\frac{\lambda_{pb}}{\lambda_b}$ for these two boundary conditions.
- The parameter λ_{t_u} that represents the nonlinearity in the axial displacement, is directly proportional to the ratio of $\left(\frac{b}{r}\right)^4$ and inversely proportional to SR^2, whereas the parameter λ_{t_w} that represents the nonlinearity in deflection, is directly proportional to $\left(\frac{b}{r}\right)^2$ and is independent of ***SR*** of the column.
- This is an important observation, which gives an insight into the effect of considering the general Green nonlinearity that for very slender columns the ratio of $\frac{\lambda_{pb}}{\lambda_b}$ will be the same as that obtained by von-Karman nonlinearity, as the value of the parameter λ_{t_u} is very much lower than the value of the parameter λ_{t_w} for a specified value of $\frac{b}{r}$. On the contrary, when the columns are short, both the values of the parameters λ_{t_u} and λ_{t_w} are of the same order and the ratio $\frac{\lambda_{pb}}{\lambda_b}$, has to be evaluated by considering the general Green nonlinearity.
- When the value of SR is small, say, 30.0 or 40.0, the effect of shear deformation is to be considered, to evaluate λ_b or $\frac{\lambda_{pb}}{\lambda_b}$. However, the values of λ_{t_u} and λ_{t_w} are independent of this effect. As such, for very short columns it is sufficient to know the value of λ_b , to predict thermal postbuckling of columns.
- As per the general behavior observed by the authors work on thermal postbuckling of columns and plates, the ratio $\frac{\lambda_{pb}}{\lambda_b}$ decreases as the over all stiffness of the column increases, for a specified ratio of $\frac{b}{r}$ considered, irrespective of the value of ***SR***, in the present investigation. It can be seen from Table-1, based on thermal buckling load parameter λ_b , the overall stiffness of the column increases for the boundary condi-

tions from hinged-hinged, clamped-hinged and clamped-clamped columns.

The proposed novel FE formulation is general and can be applied to columns with complicating effects, and to other structural members, where linear FE buckling analysis is valid.

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