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THERMAL POSTBUCKLING ANALYSIS OF THIN UNIFORM SQUARE PLATES ON WINKLER FOUNDATION - EFFECT OF USE OF GREEN'S STRAIN-DISPLACEMENT RELATIONS

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Abstract

A new simple formulation is developed in this paper, to predict the realistic thermal postbuckling behavior of the square plates, on Winkler foundation. If the plate is subjected to a uniform temperature rise, and also undergoes large deflections, mechanical equivalent inplane compressive, and tensile loads are developed, with the condition of the inplane immovability of the normal edge displacements of the plate. The nature of the distribution of these inplane compressive and tensile loads are similar, but have different magnitudes, which act along the x- and y- directions of the plate. The use of the Green's nonlinear strain-displacements relations, which do not impose any restriction on the magnitude of the large deflections, to predict the realistic thermal postbuckling loads. The thermal postbuckling results of the plates obtained from this formulation, are validated indirectly, as the corresponding results are not available in the literature, with those of the columns without the elastic foundation. For the non-zero values of the foundation parameter, the present numerical results of the plates, with respect to the central deflection, show the proper physical trends of the nonlinearity, and demonstrate the simplicity of the present formulation.

***Keywords**: Thermal buckling; Thermal post-buckling; Square plate; Column; Winkler foundation; Green's strain-displacement relations*

Notation

a = Length of the sides of the square plate

A = Area of cross-section of the column

b = Central deflection of the square plate

C = Inplane rigidity of the plate, $\dfrac{E\,t}{(1 - \nu^2)}$

D = Flexural rigidity of the plate, $\dfrac{E\,t^3}{12\,(1 - \nu^2)}$

E = Young's modulus of the material of the plate or column

K = Stiffness per unit area of the elastic (Winkler) foundation

L = Length of the column

P = Bi-axial compressive load per unit length produced in the plate due to ΔT

P_b = Thermal buckling load per unit length of the plate; thermal buckling load of the column

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P_c = Axial compressive load produced in the column due to ΔT
P_{pb} = Thermal postbuckling load per unit length of the plate; thermal postbuckling load of the column
P_x = Compressive load per unit length produced in the x- direction of the plate due to ΔT
P_y = Compressive load per unit length produced in the y- direction of the plate due to ΔT
r = Radius of gyration of the cross-section of the column
SR = Slenderness ratio of the column, $\frac{L}{r}$
t = Thickness of the plate
u = Inplane displacement of the plate in the x- direction or axial displacement of the column
v = Inplane displacements in the y- direction of the plate
ΔT = Temperature rise
T_o = Stress free temperature
T_{x_u} = Tensile load per unit length induced in the x- direction due to the nonlinearity in u; also denoted as T_u per unit length
T_{y_v} = Tensile load per unit length induced in the y- direction due to the nonlinearity in v; also denoted as T_v per unit length
w = Lateral deflection of the square plate or the column
w_x = Admissible function for w in the x- direction of the plate
w_y = Admissible function for w in the y- direction of the plate
α = coefficient of linear thermal expansion
T_{x_w} = Tensile load per unit length induced in the x- direction due to the nonlinearity in w; also denoted as T_w per unit length
T_{y_w} = Tensile load per unit length induced in the y- direction due to the nonlinearity in w; also denoted as T_w per unit length
ε_x , ε_y = Inplane strains in x- and y- directions of the plate, respectively
v = Poisson's ratio of the material of the plate or the column

Introduction

Simple analytical formulas are derived in this paper, to predict the thermal buckling and postbuckling behavior, of the thin uniform simply supported (*s-s-s-s*) or clamped (*c-c-c-c*), square plates on the elastic (Winkler) foundation [1]. The terminology of *s-s-s-s* or *c-c-c-c* used here, means that all the edges of the square plate are either simply supported or clamped, respectively. The postbuckling load is an important input to the structural engineers/researchers, working in many fields of engineering, with the aim of obtaining an efficient design of the plates, subjected to high temperature from the stress free temperature. When the normal inplane edge displacements of the plate, are immovable, the mechanical equivalent of the compressive loads, in the x- and y- directions, due to the high temperature, are developed in the plate. At the same time, if the plate undergoes large deflections, the inplane tensile loads are induced, in the x- and y- directions. For the plates, the nature of the distribution of these compressive mechanical and the tensile loads are similar but differ in the magnitude. In the square plate, the compressive mechanical or tensile loads are equal in the x- or y- directions. The mechanical/thermal postbuckling phenomenon of the plate is discussed in the works of Timoshenko and Gere [2], Ziegler and Rammerstorfer [3], Rao et al.[4], and Rao and Raju [5].

It is well known that the plate, when subjected to the critical mechanical/thermal load, suddenly collapses, and this load is the buckling load, which is obtained by using the inplane linear strain-displacement relations. Contrary to the popular belief of the engineering community that the premature structural failure of the plate takes place, even before reaching the failure strength of the material of the plate, at the onset of buckling. However, the plates still have some additional load carrying capacity above its buckling load, and the sum of the additional and the buckling loads is denoted as the mechanical/thermal postbuckling load. This additional load arises, because of the inplane stretching of the plate, due to the large lateral deflections, wherein the inplane strain-displacements contain not only the linear terms $\frac{du}{dx}$ and $\frac{dv}{dy}$, but also the non-linear terms, $\frac{1}{2}\left(\frac{du}{dx}\right)^2$ and $\frac{1}{2}\left(\frac{dv}{dy}\right)^2$ corresponding to the inplane displacements u and v, and the nonlinear terms $\frac{1}{2}\left(\frac{dw_x}{dx}\right)^2$ and $\frac{1}{2}\left(\frac{dw_y}{dy}\right)^2$ corresponding to the lateral deflection w. The subscripts x- and y- used for the lateral deflection w, represent the directions of the Cartesian coordinate system of the plate, respectively. In the theoretical analysis, to obtain the mechanical/thermal buckling loads of the plate, it is sufficient to consider only the linear terms in the strain-displacement relations. However, the mechanical/thermal postbuckling behavior of the plate is obtained, by using the Green's nonlinear strain-displacement relations.

Many researchers [3 to 8] contributed, to the study of the mechanical/thermal postbuckling of plates, by considering the simpler von-Karman nonlinear strain-displacement relations, which are applicable when deflections are moderately large. In this simpler von-Karman nonlinear theory, the nonlinear inplane displacements are ignored compared to the nonlinear lateral deflection, based on the relative magnitudes. Whereas, both the nonlinearities in the inplane displacements and the lateral deflections, which appear in the Green's nonlinear strain-displacement relations, have the same order of magnitude, and the use of which give a realistic solution of the mechanical/thermal postbuckling problem of the plates. The present simple formulas for the thermal postbuckling of square plates are derived by using the Green's nonlinearity, which is valid irrespective of the magnitude of the lateral deflection. However, care has to be taken that the large postbuckling lateral deflections of the plates (and also of the other structural members), do not interfere with the functional requirements of the structural systems.

It is to be noted that a little work is available in the literature, to study the mechanical/thermal postbuckling behavior of the plates, by using the Green's nonlinear strain-displacement relations. It is also shown in the available work on the topic of the mechanical/thermal postbuckling, that the thermal postbuckling gives the additional compressive load carrying capacity, by an order of magnitude higher when compared to the same in the mechanical postbuckling [7, 9]. Hence, in the present study, the emphasis is given on the derivation of the simple analytical formulas to evaluate the realistic thermal postbuckling load of the plates, on the Winkler foundation, by using the Green's nonlinear strain-displacement relations, which are valid even for the large deflections. These simple formulas can directly be used with ease, without going for the complex mathematical analysis, by the practicing engineers and researchers, for the specified central deflection of the square plate. The effect of elastic foundation, in terms of the foundation parameter, on the thermal postbuckling behavior of the square plate, is clearly brought out, from the numerical results of thermal postbuckling of the square plates, presented in this paper.

Green's Nonlinear Strain-Displacement Relations

In the present formulation, the Green's nonlinear strain-displacement relations, applicable to the thin plates are expressed [10], in the Cartesian x- and y- coordinate system, as

$$\varepsilon_x = \frac{du}{dx} + \frac{1}{2}\left[\left(\frac{dw_x}{dx}\right)^2 + \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2\right] \tag{1}$$

and

$$\varepsilon_y = \frac{dv}{dy} + \frac{1}{2}\left[\left(\frac{dw_y}{dy}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{dv}{dy}\right)^2\right] \tag{2}$$

For the square plate considered in this study, the expressions for the inplane displacements u and v, in the x- and y- directions are similar. Fig.1 shows a schematic diagram of a square plate, where the Cartesian x- and y- coordinate system is used, on the elastic foundation. The edges are denoted by (1), (2), (3) and (4), where the two edges(1) and (2) are opposite and the two other opposite edges (3) and (4) are opposite, as shown in Fig.1. The lateral boundary conditions of the plate are, either all the edges simply supported (*s-s-s-s*) or clamped (*c-c-c-c*), where 's' and 'c' denote simply supported and clamped edges. The boundary conditions on the inplane displacements are, $u = 0$ on the edges (1) and (2), $v = 0$ on the edges (3) and (4) of the inplane boundary conditions, as shown in Fig.1. The derivatives $\frac{dv}{dx}$ and $\frac{du}{dy}$ become zero, as the inplane displacements v and u that arise due to the Poisson's effect, are constants along the x- and y- axes, respectively. Consequently, the Green's nonlinear strain-displacement relations given in Eqs.(1) and (2) are simplified, as

$$\varepsilon_x = \frac{du}{dx} + \frac{1}{2}\left[\left(\frac{dw_x}{dx}\right)^2 + \left(\frac{du}{dx}\right)^2\right] \tag{3}$$

and

$$\varepsilon_y = \frac{dv}{dy} + \frac{1}{2}\left[\left(\frac{dw_y}{dy}\right)^2 + \left(\frac{dv}{dy}\right)^2\right] \tag{4}$$

Equations (3) and (4), are further simplified, by neglecting the nonlinear terms $\left(\frac{du}{dx}\right)^2$ and $\left(\frac{dv}{dy}\right)^2$, based on the relative magnitudes, when compared to the magnitudes of the nonlinear terms $\left(\frac{dw_x}{dx}\right)^2$ and $\left(\frac{dw_y}{dy}\right)^2$. As such, Eqs.(3) and (4) can be written, as

$$\varepsilon_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw_x}{dx}\right)^2 \tag{5}$$

and

$$\varepsilon_y = \frac{dv}{dy} + \frac{1}{2}\left(\frac{dw_y}{dy}\right)^2 \tag{6}$$

These Eqs.(5) and (6), are popularly known as the von-Karman nonlinear strain-displacement relations. Because of the simplicity of these nonlinear strain-displacement relations, many geometric nonlinear formulations are developed, as mentioned earlier. However, the von-Karman geometric nonlinear formulations have a limitation that these relations are applicable, when the lateral deflections are moderately large, as the nonlinear terms containing the inplane terms are neglected. However, if the lateral deflections are large and do not fall under the category of 'moderately large', and to predict the realistic thermal postbuckling behavior, the Green's nonlinear strain-displacements, given in Eqs.(3) and (4), have to be used.

Equivalent Mechanical Bi-Axial Compressive Loads Due to Temperature Rise

The inplane bi-axial mechanical equivalent of the compressive loads developed in the square plate, due to the immovable inplane normal edge displacements. These uniform compressive loads P_x and P_y per unit length are evaluated, by using the linear inplane strain-displacement relations, of the two-dimensional problems in both the *x*- and *y*- directions, respectively, which are given by

$$\varepsilon_x = \frac{\partial u}{\partial x} - \alpha\,\Delta T \tag{7}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} - \alpha\,\Delta T \tag{8}$$

Equations (7) and (8) are applicable if both the mechanical, and thermal strains produced in the square plate, when the temperature rise ΔT from its stress free temperature. From the conditions imposed on the inplane displacements, as the normal edge inplane displacements of the square plate are not allowed to move in its plane, and since in the present study, as the temperature rise ΔT only is considered, and the applied mechanical loads are zero, the inplane displacement field for $u\,(x, y)$ or $v\,(x, y)$ is a null field. As a result, the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ in Eqs.(7) and (8), become zeros and in the following Eqs.(9), (10), (12) and (13), the terms containing these partial derivatives do not exist for the uniform thin square plate. By following the similar argument for the uniform columns, which are much simpler one-dimensional problems, when compared to the two-dimensional square plates. The end axial displacements of the columns are restrained to move axially, and consequently the derivative $\frac{du}{dx}$ becomes zero, as such the term $\frac{du}{dx}$ does not exist in the following Eq.(14) applicable for the uniform columns.

It is to be noted that the sign convention followed in this paper is that all the compressive inplane loads are treated as positive and a negative sign is not explicitly included in the equations representing these compressive loads. The negative sign is implicitly included in all the compressive loads and is not shown in the equations that give the compressive loads.

Based on the aforementioned explanation, the compressive loads P_x and P_y , in the *x*- and *y*- directions, are developed in the square plate, due to the temperature rise ΔT from the stress free temperature T_o, obtained from Eqs.(7) and (8), can be written, as

$$P_x = \frac{E\,t}{(1-\nu^2)}\left(\varepsilon_x + \nu\,\varepsilon_y\right) \tag{9}$$

and

$$P_y = \frac{E\,t}{(1-\nu^2)}\left(\varepsilon_y + \nu\,\varepsilon_x\right) \tag{10}$$

Because of the symmetries involved in the square plate, the following relation is obtained, as

$$P_x = P_y = P \tag{11}$$

By substituting Eqs.(7) and (8), in Eqs.(9) and (10), the following equation is obtained, as

$$P_y = P_y = \frac{E\,t}{(1-\nu^2)}\,(1+\nu)\,\alpha\,\Delta T \tag{12}$$

where the loads P_x and P_y are uniform compressive loads. From Eq.(12), which is applicable for the square plates the following expression for the uniform compressive load P per unit length, which is the same, in the x- and y- directions, is obtained by

$$P = \frac{E\, t\, \alpha\, \Delta T}{(1 - \nu)} \tag{13}$$

For the uniform column, which is used for the purpose of the validation of the present formulation, the mechanical equivalent of the axial compressive load P_c, is developed due to the temperature rise ΔT from its stress free temperature T_o. The expression for the compressive load P_c is directly obtained from Eq.(7), as

$$P_c = E\, A\, \alpha\, \Delta T \tag{14}$$

Admissible Functions for Lateral Deflection of Square Plates

The lateral deflection w in the x- and y- directions are given by the exact admissible trigonometric functions, given by Leissa [11], as

$$w = b \sin\left(\frac{\Pi x}{a}\right) \sin\left(\frac{\Pi y}{a}\right) \tag{15}$$

for the *s-s-s-s* square plate, and

$$w = \frac{b}{4}\left(1 - \cos\frac{2\,\Pi\, x}{a}\right)\left(1 - \cos\frac{2\,\Pi\, y}{a}\right) \tag{16}$$

for the *c-c-c-c* square plate, respectively.

The components of the lateral deflection w in the x- and y- directions, can be written independently, without loss of rigor, by following Rao et al. [4], as

$$w_x = b \sin\left(\frac{\Pi x}{a}\right) \tag{17}$$

in the x- direction, and

$$w_y = b \sin\left(\frac{\Pi y}{a}\right) \tag{18}$$

in the y-direction, for the *s-s-s-s* square plate, and

$$w_x = \frac{b}{2}\left(1 - \cos\frac{2\,\Pi\, x}{a}\right) \tag{19}$$

in the x- direction, and

$$w_y = \frac{b}{2}\left(1 - \cos\frac{2\,\Pi\, y}{a}\right) \tag{20}$$

in the y- direction, for the *c-c-c-c* square plate, respectively. It is interesting to note that in the present simple formulation, the admissible functions for the inplane displacements are not required, as the formulation is based on the tensile loads induced in the x- and y- directions, because of the nonlinearities involved corresponding to the derivatives of the inplane displacements u and v, which can be expressed in terms of the derivatives of the lateral deflection w of the square plate, as given by Dym [9].

Logical Steps to Evaluate Thermal Postbuckling Load

If the square plate is subjected to a uniform temperature rise, a bi-axial mechanical equivalent of the compressive load P per unit length is developed, which is the same in both the x- and y- directions. If the temperature increases further, the compressive load also increases, and at a particular temperature, which is the critical temperature, this bi-axial compressive load reaches the thermal buckling load P_b. Further increase of the temperature above the critical temperature, the square plate starts to undergo the lateral deflection w. If the increase in the temperature is much higher than the critical temperature, then the magnitude of the lateral deflection becomes significantly larger. Consequently, the use of the Green's nonlinear strain-displacement relations to obtain the realistic thermal postbuckling behavior of the square plate contains the nonlinear terms, in the lateral deflection w and the inplane displacements u (or v), in both the x- and y- directions, apart from the usual linear terms, respectively. These nonlinear terms induce the inplane tensile loads T_{x_w} and T_{x_u} in the x- direction, and T_{y_w} and T_{y_v} per unit length in the y- direction, as a result of the inplane stretching of the plate, because of the large lateral deflection w. In the present study, because of the symmetries existing in the square plate, these tensile loads can be written as $T_{x_w} = T_{y_w} = T_w$ and $T_{x_u} = T_{y_v} = T_u = T_v = T_{u,v}$. As such, the derivation of the ratio of the thermal postbuckling and buckling loads presented here are the same, by considering either the x- or y- directions of the square plate, respectively. Hence, in the formulation the tensile loads T_w and T_u are derived corresponding to the x- direction only, without using the suffix 'x' explicitly, for the sake of simplicity in writing the tensile loads T_w and T_u, including the bi-axial compressive load P. It is to be noted that the induced tensile loads T_w and T_u because of the inplane stretching, due to large lateral deflection increases the thermal compressive load carrying capacity of the plate, beyond its thermal buckling load P_b. The increased compressive load is the thermal postbuckling load P_{pb}, of the square on the elastic foundation.

It is to be noted that the tensile loads T_w and T_u are independent, and the thermal postbuckling and buckling loads P_{pb} and P_b are dependent on the elastic foundation, where the stiffness of the elastic foundation is K per unit area. Because of the nature of the tensile loads T_w and T_u, the total tensile load induced in the square plate is $(T_w + T_u)$. The relation between the thermal postbuckling and buckling loads P_{pb} and P_b, and the total tensile load $(T_w + T_u)$, can be written, as

$$P_{pb} = P_b + T_w + T_u \tag{21}$$

In the nondimensional form, Eq.(21) can be written, as

$$\lambda_{pb} = \lambda_b + \lambda_{T_w} + \lambda_{T_u} \tag{22}$$

or

$$\frac{\lambda_{pb}}{\lambda_b} = 1 + \frac{\lambda_{T_w}}{\lambda_b} + \frac{\lambda_{T_u}}{\lambda_b} \tag{23}$$

The nondimensional parameters given in Eq.(23) are defined, as

$$\lambda_{pb} \left(= \frac{P_{pb}\, a^2}{\pi^2 D} \right), \ \lambda_b \left(= \frac{P_b\, a^2}{\pi^2 D} \right),$$

$$\lambda_{T_w} \left(= \frac{T_w\, a^2}{\pi^2 D} \right) \text{ and } \lambda_{T_u} \left(= \frac{T_u\, a^2}{\pi^2 D} \right).$$

The tensile load parameters λ_{T_w} and λ_{T_u} do not dependent on the elastic foundation parameter $\gamma \left(= \frac{K a^4}{\pi^4 D} \right)$ and the thermal buckling load parameter λ_b depend on the elastic foundation parameter λ.

Evaluation of Tensile Loads Induced Due to Inplane Stretching

The tensile loads T_w (or λ_{T_w}) and T_u (or λ_{T_u}), which are induced in the square plate, corresponding to the nonlinear terms in the lateral deflection w and the inplane displacement u, because of the use of the Green's nonlinear strain-displacement relations, are obtained using the following procedure. If the normal inplane edge displacement at $x = 0$ of the square plate is immovable, and the condition of the immovability on the inplane normal edge displacement at the opposite edge $x = a$ is relaxed, the inplane normal outward (positive x-direction) displacement u_o of the edge at $x = a$, which is developed due to an applied inplane uniform tensile load T_w per unit length acting at the edge $x = a$, can be obtained [12], as

$$u_o = \frac{T\, a}{t\, E} \tag{24}$$

The inplane normal edge inward (negative x- direction) displacement u_i at $x = a$, due to the large deflection w is obtained, following the procedure given by Woinowsky-Krieger [13], as

$$u_i = \frac{1}{2} \int_0^a \left(\frac{dw}{dx} \right)^2 dx \tag{25}$$

To make the edge $x = a$ immovable, which is the initial condition that is specified on the immovability of the normal edge inplane edge displacements, the relation $u_i = u_o$ on the magnitudes of u_o and u_i is used. Then, the tensile load T_w induced in the square plate due to the inplane stretching caused because of the large lateral deflection w, as

$$T_w = \frac{E\, t}{2\, a} \int_0^a \left(\frac{dw}{dx} \right)^2 dx \tag{26}$$

Following a similar procedure, used to evaluate T_w as given in Eq.(26), the tensile load T_u per unit length induced in the square plate due to the nonlinear inplane term corresponding to u, is evaluated by using the relation $\frac{du}{dx} = \frac{1}{2} \left(\frac{dw}{dx} \right)^2$ given by Dym [9], as

$$T_u = \frac{E\, t}{8\, a} \int_0^a \left(\frac{dw}{dx} \right)^4 dx \tag{27}$$

It is to be noted that in the present simple formulation, which is based on the uniform tensile loads induced in the plate due to large lateral deflections, an explicit evaluation of the inplane displacement u or v, where $u = v$ for the square plate, is not required [6]. The evaluation of the inplane displacement field of u (or v) is difficult, if not impossible [8], when compared to choosing the lateral deflection field of w, which is readily available [11]. Hence, in the present simple formulation, the uniform tensile load T_u or T_v, where ($T_u = T_v$) for the square plate, and the nonlinear terms $\left(\frac{du}{dx}\right)^2$ or $\left(\frac{dv}{dy}\right)^2$, where $\left\{\left(\frac{du}{dx}\right)^2 = \left(\frac{dv}{dy}\right)^2\right\}$, for the square plate, are evaluated by using the relation $\frac{du}{dx} = \frac{1}{2}\left(\frac{dw}{dx}\right)^2$ or $\frac{dv}{dy} = \frac{1}{2}\left(\frac{dw}{dy}\right)^2$ given by Dym [9], is used to obtain the nonlinear term as $\left(\frac{du}{dx}\right)^2$ as $\frac{1}{4}\left(\frac{dw}{dx}\right)^2$ or $\left(\frac{dv}{dy}\right)^2$ as $\frac{1}{4}\left(\frac{dw}{dy}\right)^2$, and as such the expression for the uniform tensile load T_u in Eq.(27) is obtained, by using the admissible function chosen for the lateral deflection w. In a similar way the uniform tensile load T_v can be evaluated by replacing the integrand $\left(\frac{dw}{dx}\right)^4$ by $\left(\frac{dw}{dy}\right)^2$ in Eq.(27). Because of the symmetries involved in the square plate, all the quantities corresponding to the x- direction of the plate only are considered in the present study, and even if the y- direction is considered in the analysis, the required quantities will be the same, obtained by considering the x- direction.

Ratio of Thermal Postbuckling to Buckling Loads

The integrals involved in the expressions for the tensile loads T_w and T_u per unit length are evaluated, from the admissible functions taken for the lateral deflection w. These functions, as mentioned earlier, are $b \sin\left(\frac{\Pi x}{a}\right)$ and $\frac{b}{2}\left(1 - \cos\frac{2\,\Pi\,x}{a}\right)$ for the *s-s-s-s* and *c-c-c-c* denote the simply supported and clamped edges of the square plates, respectively. The important requirement of these functions is that the central deflection of the square plate, at $x = y = \frac{a}{2}$ is b. The ratio of $\frac{\lambda_{pb}}{\lambda_b}$ for both the *s-s-s-s* and the *c-c-c-c* square plates is the same as given, by

$$\frac{\lambda_{pb}}{\lambda_b} = 1 + \frac{2.73}{\lambda_b}\left(\frac{b}{t}\right)^2 + \frac{5.052}{\lambda_b\, n^2}\left(\frac{b}{t}\right)^4 \tag{28}$$

The difference of the ratio $\frac{\lambda_{pb}}{\lambda_b}$ between the *s-s-s-s* and the *c-c-c-c* square plates is due to the buckling load parameters λ_b, which is evaluated, considering the elastic foundation parameter γ. Eq.(28) gives the ratios of $\frac{\lambda_{pb}}{\lambda_b}$, for different values of the central lateral deflection to the thickness ratio $\frac{b}{t}$ and $n\left(= \frac{a}{t}\right)$, for the square plates considered in the present study.

Validation of Present Formulation

For the simple formulation proposed in this paper, it is necessary to validate the numerical results in terms the ratios of the thermal postbuckling and the buckling loads $\frac{\lambda_{pb}}{\lambda_b}$ of the square plate, considering the nonlinearities involved in the inplane displacements and the lateral deflection, by using the Green's nonlinear strain-displacement relations. In the absence of any similar numerical results in the literature, for the *s-s-s-s* and *c-c-c-c* square plates, to the best of the authors' knowledge, it is not possible to directly validate the numerical results that are obtained from the simple formulas derived. Hence, an indirect method is used, to validate the present simple formulation, in which the bending of the s-f-s-f, square plate, as shown in Fig.2. The square plate, considered for the purpose of validation, bends in the x- direction only, where '*f*' represents the free edge when the bending only is considered, without the elastic foundation ($\gamma = 0$). It is to be noted that the boundary conditions, on the inplane displacements u and v are the same as taken for the *s-s-s-s* and *c-c-c-c* plates. The square plate can also be analyzed as a uniform *h-h* or *c-c* column, where '*h*' and '*c*' denote the hinged and clamped boundary conditions of the column, by suitably defining the nondimensional parameters, involved in the *h-h* and *c-c* columns and the *s-f-s-f* and *c-f-c-f* square plates. As the numerical results for the thermal postbuckling are available, from the recent unique formulation of Shirong and Changjun [15] for the uniform column, and since a good agreement of the present numerical results of the *s-f-s-f* and *c-f-c-f* square plates are in good agreement with those of the *h-h* and *c-c* uniform columns

[15], the present numerical results, in the form of $\frac{\lambda_{pb}}{\lambda_b}$ obtained for the *s-s-s-s* and *c-c-c-c* square plates, are deemed to have been validated. Hence, the present simple procedure proposed in this paper, is one of the elegant indirect methods, to validate the numerical results obtained, for the thermal postbuckling in the form of $\frac{\lambda_{pb}}{\lambda_b}$ of the *s-s-s-s* and *c-c-c-c* square plates considered here, in terms of the central deflection ratio $\frac{b}{t}$.

To validate the proposed analytical formula for the thermal postbuckling behavior of the square plate with uniaxial loading, in the *x*- direction only, is considered. The thermal compressive loading acts along the two opposite edges of the square plate, which are restrained to move in the *x*- direction of the plate. On the other two edges of the square plate, there are no geometric inplane boundary conditions in the *y*- direction. The Green's strain-displacement relations, applicable for the plates, are used to validate the proposed formulation.

Due to the large lateral deflection, the tensile loads developed in the square plate T_w and T_u, by equating the outward and inward axial displacements in the *x*- direction, following the procedure of Woinowsky-Krieger [13], as

$$T_w = \frac{E\,t}{2\,a}\int_0^a \left[\frac{dw_x}{dx}\right]^2 dx \tag{29}$$

$$T_u = \frac{E\,t}{8\,a}\int_0^a \left[\frac{dw_x}{dx}\right]^4 dx \tag{30}$$

The one term trigonometric admissible function for the lateral deflection *w* in the *x*- direction only is used [11], by taking in to account of the boundary conditions on the inplane displacements in the general formulation, as

$$w_x = b\,\sin\left(\frac{\Pi\,x}{a}\right) \tag{31}$$

for the *s-f-s-f* and

$$w_x = \frac{b}{2}\left(1 - \cos\frac{2\,\Pi\,x}{a}\right) \tag{32}$$

for the *c-f-c-f* square plates, respectively.

The tensile load parameters due to the large deflection, are expressed in the nondimensional form, as $\lambda_{T_w} = \frac{T_w\,a^2}{\Pi^2 D}$ and $\lambda_{T_u} = \frac{T_u\,a^2}{\Pi^2 D}$, by using the relation $\frac{du}{dx} = \frac{1}{2}\left(\frac{dw_x}{dx}\right)^2$ [9]. The ratio of $\frac{\lambda_{pb}}{\lambda_b}$ can be written following the general formulation, as

$$\frac{\lambda_{pb}}{\lambda_b} = 1 + \frac{\lambda_{T_w}}{\lambda_b} + \frac{\lambda_{T_u}}{\lambda_b} \tag{33}$$

In Eq.(33) the buckling load parameters used are corresponding to the *h-h* or *c-c* columns [2], without the elastic foundation $(\gamma = 0)$ and the ratios of $\frac{\lambda_{pb}}{\lambda_b}$ are obtained, as

$$\frac{\lambda_{pb}}{\lambda_b} = 1 + 3\left(\frac{b}{t}\right)^2 + \frac{9\,\Pi^2}{16\,n^2}\left(\frac{b}{t}\right)^4 \tag{34}$$

for the *s-f-s-f*, and

$$\frac{\lambda_{pb}}{\lambda_b} = 1 + \frac{3}{4}\left(\frac{b}{t}\right)^2 + \frac{9\,\Pi^2}{64\,n^2}\left(\frac{b}{t}\right)^4 \tag{35}$$

for the *c-f-c-f* square plates respectively.

The Eqs.(34) and (35) can be reduced to the column problem, by using the appropriate equivalent terms as given, by $a = L$, $t = 2\sqrt{3}\,r$ and $n = \frac{SR}{2\sqrt{3}}$, where *SR* is the slenderness ratio $\frac{L}{r}$. Based on these equivalent terms Eqs.(34) and (35) reduce to

$$\frac{\lambda_{pb}}{\lambda_b} = 1 + \frac{1}{4}\left(\frac{b}{r}\right)^2 + \frac{3\,\Pi^2}{64\,SR^2}\left(\frac{b}{r}\right)^4 \tag{36}$$

for the *h-h* column, and

$$\frac{\lambda_{pb}}{\lambda_b} = 1 + \frac{1}{16}\left(\frac{b}{r}\right)^2 + \frac{3\,\Pi^2}{256\,SR^2}\left(\frac{b}{r}\right)^4 \tag{37}$$

for the *c-c* column, respectively.

Numerical Results and Discussion

The square plate resting on the elastic foundation with the foundation stiffness K per unit area, and the boundary conditions for both the inplane displacements u and v, and the lateral deflection w on the edges (1), (2), (3) and (4), are shown in Fig.1. The two opposite edges (1) and (2) are perpendicular to the x- axis and the other two opposite edges (3) and (4) are perpendicular to the y- axis of the square plate. The immovability condition of the inplane displacements (Fig.1) are used to evaluate the expressions for the tensile loads T_w and T_u induced in the square plate, because of the inplane stretching, due to the large deflections of the plate. These tensile loads, used in this study, are evaluated in the x- direction only, due to the symmetries existing in the square plate.

The thermal buckling load parameters λ_b of the thin plate on the elastic foundation [4], which is required to obtain the ratios of the thermal postbuckling and the buckling load parameters $\frac{\lambda_{pb}}{\lambda_b}$, for several values of the elastic foundation parameters γ that vary from 0.0, 1.0, 2.0 and 5.0, are presented in Table-1. For these values of elastic foundation parameters considered, the phenomenon of changing mode shapes of buckling do not exist for the *s-s-s-s* and *c-c-c-c* thin square plates [2]. These values of λ_b are given for quickly evaluating the values of the thermal postbuckling results of the square plate on elastic foundation, in terms of $\frac{\lambda_{pb}}{\lambda_b}$, by using the tensile load parameters λ_{T_w} and λ_{T_u}, the evaluation of which has been explained in the earlier sections.

It is to be noted that the most important aspect of using the Green's nonlinear strain-displacement relations, to obtain the ratios of the thermal postbuckling to buckling loads $\frac{\lambda_{pb}}{\lambda_b}$ of the thin plates (n = 34.6410 for the *s-f-s*-f and n = 46.1880 for the *c-f-c-f*), are not available in the literature, to the best of the authors' knowledge. However, if the plate bends in the x- direction only, the *s-f-s-f* or the *c-f-c-f* plate can be treated as the slender *h-h* column (*SR* = 120) or the *c-c* column (*SR* = 160). It is to be noted that the values of n taken for the *s-f-s-f* and *c-f-c-f* square plates correspond to values of the *SR* of the *h-h* and *c-c* columns, respectively. The values $\frac{\lambda_{pb}}{\lambda_b}$ of these slender uniform columns are readily available in the unique work of Shirong and Changjun [15]. The present thermal postbuckling results $\frac{\lambda_{pb}}{\lambda_b}$ obtained for the thin square plates that bend in the x- direction only and with those presented for the columns [15] match well, for the boundary conditions considered. These numerical results, that are presented in Tables-2 and 3, indicate the effectiveness of the indirect way of validation of the present numerical results. Further, it can also be observed from these two tables, that by properly defining the nondimensional central deflection ratio $\frac{b}{t}$ for the plates and the similar ratio $\frac{b}{r}$ for the columns, the indirect way of validation is successful, and gives the same results for the ratio of $\frac{\lambda_{pb}}{\lambda_b}$.

Table-1 : Thermal Buckling Load Parameters λ_b of Square Plate for Various Values of Elastic Foundation Parameters [4]				
Boundary Conditions	λ_f=0.0	λ_f=1.0	λ_f=2.0	λ_f=5.0
s-s-s-s	2.0000	2.4860	2.9874	4.4913
c-c-c-c	5.3333	5.7016	6.0819	7.2229

Table-4 gives the numerical results of the ratio $\frac{\lambda_{pb}}{\lambda_b}$ for the thin *s-s-s-s* square plates (n = $\frac{a}{t}$ = 34.6410) on the elastic foundation. The values of the foundation parameter γ are taken as 0.0, 1.0, 2.0 and 5.0. For these values of the foundation parameter, the transition of the buckling mode shapes do not exist, as deduced from the study of Timoshenko and Gere [2], for the thin square plates. It can also be observed from this table that the values of $\frac{\lambda_{pb}}{\lambda_b}$, decrease with the foundation parameter, irrespective of the value of the central deflection ratio $\frac{b}{t}$. In the absence of the numerical results for this problem, the physical trends of $\frac{\lambda_{pb}}{\lambda_b}$, namely, the nonlinearity in the ratio of $\frac{\lambda_{pb}}{\lambda_b}$ increases, with the increase in the ratio of $\frac{b}{t}$. And due to the increase of the foundation parameter, the stiffness of the plate increases, and hence, decreases the ratio of $\frac{\lambda_{pb}}{\lambda_b}$.

Table-2 : Validation of $\frac{\lambda_{pb}}{\lambda_b}$ Values Obtained from Intuitive Formulation for *s-f-s-f* Square Plate With *h-h* Column

$\frac{b}{r}$	$\frac{b}{t}$	Present Study		Shirong and Changjum [15]	Absolute Value of % Difference
		Column (SR = 120)	Square Plate (n = 34.641)		
1.3329	**0.3847**	1.4441	**1.4441**	1.4438	0.0270
2.6625	**0.7685**	2.7735	**2.7735**	2.7764	0.1017
3.9857	**1.1505**	4.9789	**4.9789**	5.0019	0.4586
5.2993	**1.5297**	8.0451	**8.0451**	8.1265	1.0006
6.6005	**1.9053**	11.9516	**11.9516**	12.1590	1.7055

Table-3 : Validation of $\frac{\lambda_{pb}}{\lambda_b}$ Values Obtained from Intuitive Formulation for *c-f-c-f* Square Plate With *c-c* Column

$\frac{b}{r}$	$\frac{b}{t}$	Present Study		Shirong and Changjum [15]	Absolute Value of % Difference
		Column (SR = 160)	Square Plate (n = 46.188)		
1.6211	**0.4679**	1.1642	**1.1642**	1.1638	0.0411
3.2422	**0.9359**	1.6574	**1.6574**	1.6563	0.0718
4.8634	**1.4039**	2.4808	**2.4808**	2.4816	0.0315
6.4855	**1.8722**	3.6368	**3.6368**	3.6464	0.2619
8.1056	**2.3398**	5.1257	**5.1257**	5.1604	0.6705
9.7268	**2.8078**	6.9535	**6.9535**	7.0365	1.1781
11.347	**3.2755**	9.1220	**9.1220**	9.2914	1.8228

The present numerical results are consistent, according to the physical reasoning, that the increasing and decreasing trends of the nonlinearity of the thermal postbuckling, is observed for the thin *s-s-s-s* square plates. A similar behavior is also observed for the thin (n = 46.1880) *c-c-c-c* square plate, for which the numerical results are presented in Table-5. These numerical results are also consistent with the physical trends, as discussed for the thin *s-s-s-s* square plates.

Conclusions

Simple formulas are derived using a relatively simple formulation, to evaluate the realistic thermal postbuckling behavior of thin square plates on the elastic foundation. Though, many researchers, who used the von-Karman nonlinearity, which is applicable for the moderately large deflections, the Green's nonlinearity, which does not have any restriction on the magnitude of the lateral deflections, is employed in the present study. The simplicity of the present formulation is demonstrated, by evaluating the realistic thermal postbuckling behavior, from the nondimensional parameters, namely, the bi-axial tensile and the buckling loads only, which eliminates the complex mathematical treatment of the problem. The present simple formulation is validated, by using an indirect method. Hence, the realistic postbuckling results of the square plates are deemed to be accurate, based on the increasing and decreasing trends of the nonlinearities that effect the thermal buckling load ratios, by varying the foundation and the central deflection parameters. The authors are of the opinion that the simple formulas developed in this paper can be used with confidence, by the practicing engineers and researchers working on this topic of thermal postbuckling of the square plate, by using the Green's

Table-4 : Values of $\frac{\lambda_{pb}}{\lambda_b}$ Varying with Elastic Foundation Parameter for S-S-S-S Square Plate (n = 34.641)

$\frac{b}{t}$	Present Study			
	$\lambda_f = 0.0$	$\lambda_f = 1.0$	$\lambda_f = 2.0$	$\lambda_f = 5.0$
0.0	1.0000	1.0000	1.0000	1.0000
0.2	1.0546	1.0439	1.0365	1.0243
0.4	1.2184	1.1757	1.1462	1.0972
0.6	1.4916	1.3955	1.3291	1.2189
0.8	1.8744	1.7034	1.5854	1.3893
1.0	2.3671	2.0997	1.9152	1.6087
2.0	6.4936	5.4194	4.6777	3.4461
3.0	13.4555	11.0200	9.3383	6.5461
4.0	23.3788	19.0030	15.9815	10.9647
5.0	36.4406	29.5106	24.7256	16.7808

Table-5 : Values of $\frac{\lambda_{pb}}{\lambda_b}$ Varying with Elastic Foundation Parameter for C-C-C-C Square Plate (n = 46.188)

$\frac{b}{t}$	Present Study			
	$\lambda_f = 0.0$	$\lambda_f = 1.0$	$\lambda_f = 2.0$	$\lambda_f = 5.0$
0.0	1.0000	1.0000	1.0000	1.0000
0.2	1.0205	1.0191	1.0179	1.0151
0.4	1.0820	1.0766	1.0718	1.0604
0.6	1.1847	1.1724	1.1616	1.1360
0.8	1.3285	1.3066	1.2873	1.2419
1.0	1.5134	1.4792	1.4491	1.3782
2.0	3.0591	2.9218	2.8014	2.5168
3.0	5.6530	5.3428	5.0707	4.4276
4.0	9.3219	8.7671	8.2804	7.1303
5.0	14.1031	13.2295	12.4633	10.6523

nonlinear strain-displacement relations. The authors also postulate that for the very thin plate, where the value of n is higher, the thermal postbuckling behavior obtained is the same, irrespective of the use of Green's or von-Karman nonlinear strain-displacement relations, by critically analyzing the thermal postbuckling equation for the square plates.

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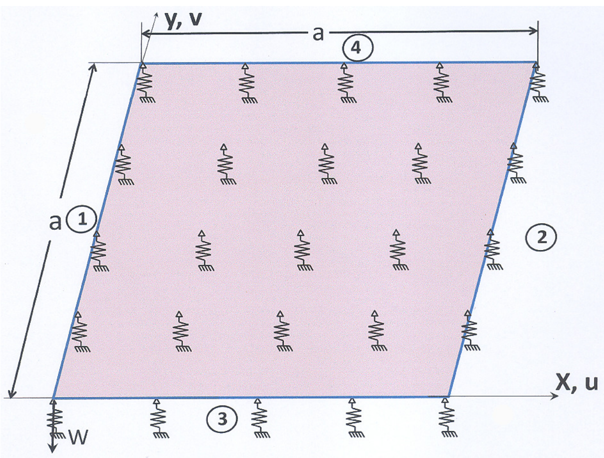

Fig.1 A Schematic Diagram of the Square Plate on Elastic (Winkler) Foundation

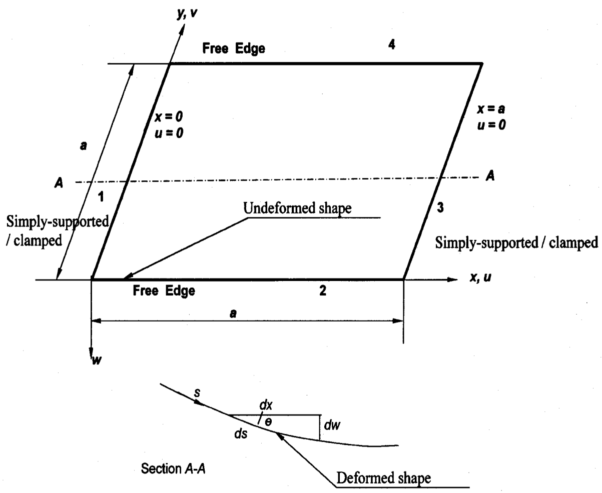

Fig.2 Square Plate with Uniaxial Compression Due to Temperature Rise for Both S-f-S-f or C-f-C-f Boundary Conditions