

VIBRATION AND BUCKLING ANALYSIS OF NANO-SCALE BEAMS VIA NONLOCAL ELASTICITY AND TIMOSHENKO BEAM THEORY: A DIFFERENTIAL QUADRATURE APPROACH

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Abstract

A single-elastic beam model is being developed based on Eringens nonlocal elasticity and Timoshenko beam theory. Nonlocal parameter takes into account the small scale effects in the analysis of nano-size structures. Derived herein is a decoupled sixth-order nonlocal governing differential equations for the frequency and stability analysis of nano-scale short beams. The effect of shear deformation is thereby included in the small scale analysis. A Differential Quadrature (DQ) approach is being applied and its elaborate method of solution is illustrated. The higher order DQ approach is found to be a good numerical technique for the convenient and rapid solution of the aforementioned nonlocal problems. Frequency and buckling results for various scale-based nonlocal parameters are shown. Effect of number of interpolation points on the accuracy of the results is also investigated. It is seen that there is a significant effect of aspect ratio and nonlocal parameter on the nonlocal frequency and buckling loads of nano-scale beams. Also, the present study could open up a new approach of solution technique for the analysis of nano-scale structures based on nonlocal Timoshenko theory.

Keywords: *Nonlocal elasticity, classical mechanics, Timoshenko beam, vibration, buckling and differential quadrature.*

Introduction

For modern advancement of aerospace technologies, the understanding and development of nanotechnology becomes indispensable. Nanotechnology promises for future aerospace applications the following assets: high strength, low weight composites, improved electronics and displays, variety of physical sensors, and multifunctional materials with embedded sensors, large surface area materials and novel filters and membranes for air purification, nanomaterials in tires and brakes, etc. Thus from both experimental and theoretical research communities, nano-scale structures such as nanobeams [1], nanorings [2], nanoribbons [3], nanoplates [4], and nanotubes [5-6] (CNTs) have gained considerable attention. The nanostructures possess much superior mechanical, electrical, electronic and thermal properties as compared to the conventional structural materials. Advanced area of novel applications of these nanotechnology based structures in aerospace field is foreseen in the coming years [7]. For example, the superior strength and low weight of fullerenes (CNTs) may open the frontier to space travel by drastically decreasing the cost of launch to orbit.

Since experiments at the nanoscale are extremely difficult and molecular dynamic (MD) simulations remain prohibitively expensive for large-sized atomic system, continuum models continue to play an essential role in the study of micro/nano scale structures. Size-dependent continuum based methods are thus getting increasing attention in modeling small sized structures as it offers much faster analyses than molecular simulations for systems of engineering interest. The most reportedly used theory for analyzing nano-scale structures is the nonlocal elasticity theory [8-9].

In nonlocal elasticity theory the stress at a point is defined as a function not only of strain at that point (classical local mechanics) but also a function of the strain at all other points in the body [9]. The contribution of forces between atoms and the effect of internal and external lengths are being included in the formulation. Recently there has been growing interest for application of nonlocal continuum mechanics especially in the field of wave propagation, fracture mechanics, dislocation mechanics, and micro/nano technologies etc.

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Nonlocal elasticity theory holds an important area of research for the future structural developments and design in modern aerospace and aeronautical field. This is due to the fact that small-size structures such as CNT, micro/nano sensors and actuators which are being applied in aerospace structures (CNT-reinforced composite [10], MEMS / NEMS devices (smart structures)) could not be accurately analysed by local (classical) theory. The local classical mechanics theory is assumed to be as scale free theory. Experiments and atomic simulation have shown that there is a significant size effect in mechanical properties when the dimensions of these structures become small. Applying nonlocal theory to such small structures could lead to correct prediction of mechanical behaviors.

The importance of nonlocal elasticity theory motivated the scientific community to explore the accurate bending, vibration, buckling, and behavior of micro/nano scale structures. Various nonlocal elasticity work are found in Peddieson et al. [11], Sudak [12], Wang et al. [13], Lu et al. [14], Reddy [15] and Pradhan and Murmu [16].

The present paper proposes a differential quadrature formulation for dynamic and stability analysis of nano-scale beams which includes Timoshenko beam theory and Eringens nonlocal variables. These formulations will be important in the application and analysis of nonlocal theories in structural studies for small-size analysis (CNT in CNT-reinforced composites). Recently Murmu and Pradhan [17] reported the use of differential quadrature method for the buckling analysis of nano-sized beams supported on Winkler foundation.

The Differential Quadrature Method (DQM) is a simple and efficient technique for solving partial differential equations as reported by Bellman and Casti [18] and Bellman et al. [19]. DQ researchers have successfully applied this method in solving various engineering problems [20-24]. Recently, Pradhan and Murmu [23] applied DQ method for vibration of functionally graded beams. In the DQM a smaller number of interpolation points are adequate to yield reasonably accurate results. This is because all uniform or non uniform interpolation points are used to represent the each-order derivation of the function at each point. Thus accurate numerical solutions are obtained by employing few interpolation points. The present numerical technique is successfully applied in the analysis of beams, plates and shells. The present authors have shown the suitability and convenience of using the DQ method for analysis of nonlocal vibration and buckling problems ([16], [17], and [25]). However the previous

works were limited to nonlocal DQ Euler-Bernoulli beam theory.

Thus, in this present work, the DQ technique is being applied to nonlocal dynamic and stability beam problems considering both nonlocal elasticity and Timoshenko beam theory. The effect of shear deformation in nano-sized beams is thereby highlighted in the analysis. Single decoupled governing sixth-order differential equation for frequency and stability analysis of nano-scale Timoshenko beams is derived using Eringens nonlocal elasticity theory. Differential Quadrature (DQ) approach is being applied and its analogous nonlocal Timoshenko DQ formulations are presented. The present higher-order DQ method is found to be a good numerical approach for the convenient and rapid solution of the nonlocal problems dealing with nanostructures. The effects of (i) number of DQ grid points, (ii) aspect ratio and (iii) nonlocal parameter on the accuracy of the nonlocal vibration and buckling results of nanobeams are investigated and discussed. Finally the use of present nonlocal Timoshenko beam model in the vibration and buckling analysis of CNT is shown.

Nonlocal Elasticity Theory

In nonlocal elasticity theory, the stress field at a point x in an elastic continuum depends on strains at all other points of the body as mentioned by Eringen [9]. This is in accordance with atomic theory of lattice dynamics and experimental observations on photon dispersion. The most general form of the constitutive equation for nonlocal elasticity involves an integral over the whole body. Thus, the nonlocal stress tensor α at a point x for a linear, homogenous body is expressed as

$$\alpha = \int_V H(|x' - x|, \phi) t(x') dX' \quad (1)$$

The terms $t(x)$ and $H(|x' - x|, \phi)$ are the classical stress at point x and the nonlocal modulus, respectively. The nonlocal modulus can be thought of attenuation function which incorporates into the constitutive equations the nonlocal effects at the reference point x produced by local strain at the source x' . $|x' - x|$ represents the distance in Euclidean form. The parameter ϕ is a material constant. The constant ϕ depends on the internal (e.g. lattice parameter, granular size, distance between C-C bonds) and external characteristics lengths (e.g. wavelength). The macroscopic stress t at a point x in a Hookean solid is related to the strain at the point by the generalized Hookes law

$$t(X) = C(X) : \varepsilon(X) \quad (2)$$

C is the fourth-order elasticity tensor; and $:$ denotes the double dot product. Eqs. (1) and (2) define the nonlocal constitutive behavior of a Hookean solid.

As solving of integral constitutive Eq. (1) is difficult, a simplified equation of differential form is used as a basis of all nonlocal constitutive formulation [9]

$$\left(1 - \phi^2 l^2 \nabla^2\right) \sigma = t, \quad \phi = \frac{e_0 a}{l} \quad (3)$$

where ∇^2 is the Laplacian operator, e_0 is a constant for adjusting the model in matching some reliable results by experiments or other models. The parameter e_0 is estimated such that the relations in Eqn (3) of the model could provide satisfied approximation of atomic dispersion curves of plane waves with those of atomic lattice dynamics. The other parameters a and l denotes the internal and external characteristic lengths, respectively. The internal characteristic lengths a include lattice parameter, granular size, or molecular diameters.

For the case of one-dimensional structures such as a small scale beam, the Laplacian operator is reduced to one dimensional form and the strains in the y and z directions are negligible. For a beam structure, the sizes in thickness and width are much smaller than the size in length. Hence a uniaxial stress state is established in the one dimensional nonlocal theory. Thus the nonlocal constitutive relation for the macroscopic stress is given as Reddy [15]

$$\sigma_{xx} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E \varepsilon_{xx} \quad (4)$$

$$\tau(x) - (e_0 a)^2 \frac{\partial^2 \tau(x)}{\partial x^2} = G \gamma(x) \quad (5)$$

where E and G are the Young's modulus and shear modulus, respectively, and γ is the shear strain. Thus, the scale coefficient ($e_0 a$) in the modeling will lead to small-scale effect on the response of structures in nano-size. In the limit when the effects of strains at points other than x are neglected, one obtains local or classical theory of elasticity from the nonlocal elasticity theory.

The importance of the term small scale coefficient or nonlocal parameter, ($e_0 a$) can be found in Murmu and

Pradhan [17]. The nonlocal parameter is defined as ($e_0 a$), ($e_0 a$)², ($e_0 a/L$) or ($e_0 a/L$)² according to the suitability of the formulations as reported in various papers. Here L is the characteristic length. The inclusion of the nonlocal scale coefficient in the above equation takes into account the effects of "scale-factor", usually smaller size. The nonlocal parameter or scale coefficient transforms the classical local equation into a nonlocal mechanics equation. Classical continuum elasticity, which is a scale free theory, cannot predict the size effects. At nanometer scales, size effects often become prominent. Both experimental and atomistic simulation results have shown a significant "size-effect" in mechanical properties when the dimensions of these structures become small. As the length scales are reduced, the influences of long-range inter-atomic and intermolecular cohesive forces on the static and dynamic properties tend to be significant and cannot be neglected. These observations are extensively documented by experimentalist. The classical theory of elasticity being the long wave limit of the atomic theory excludes these effects. Thus the traditional classical continuum mechanics would fail to capture the small scale effects when dealing in nano structures. The small size analysis using local theory over predicts the results. Thus the consideration of nonlocal scale coefficient ($e_0 a$) is necessary for correct prediction of micro/nano structures [9]. Nonlocal theory considers long-range interatomic interaction and yields results dependent on the size of a body. It is also reported in the paper of Chen et al. [26] that the nonlocal continuum theory based models are physically reasonable from the atomistic viewpoint of lattice dynamics and Molecular Dynamics (MD) simulations. It should be noted that when the nonlocal scale coefficient ($e_0 a$) is zero, the equation (4-5) reduces to that of classical mechanics one.

For a specific material or structure, the corresponding nonlocal parameter ($e_0 a$) can be estimated experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics. A value of 0.39 was used by Eringen [9] for e_0 . Sudak [12] used the length of C-C bond equal to 0.142 nm for CNT stability analysis as internal characteristic length ' a '. Wang and Hu [27] used strain gradient method to propose an estimate of the value around $e_0 a = 0.288$. Wang et al. [28] suggested a value of $e_0 a = 0.7$ nm for the application of the nonlocal theory in estimation of stiffness of CNTs based on the compression studies. Generally for the analysis of carbon nanotubes the nonlocal scale coefficients $e_0 a$ are taken in the range of 0 - 2 nm. Still contemporary research

is going on to find the exact values of nonlocal parameters for various nano level structural problems.

Research efforts are undergoing these days to bring in the scale effects within the formulation by modifying the traditional classical mechanics. These other scale based theories include micromorphic theory, microstructure theory, micropolar theory, Cosserat theory, modified couple stress theory etc.

Formulation

Nonlocal Timoshenko Beam Theory

According to Timoshenko beam theory, the displacement field at any point is written as:

$$\begin{aligned} u_1 &= u(x, t) + z \psi(x, t) \\ u_2 &= 0 \\ u_3 &= w(x, t) \end{aligned} \tag{6}$$

where, x is the longitude coordinate, z is the coordinate measured from the mid-plane of the beam, ψ is the rotation of the cross section (Fig.1). The term u and w are the axial and transverse displacements of the point $(x, 0)$ on the mid-plane (i.e., $z = 0$) of the beam. The nonzero strains according to Timoshenko beam theory are expressed as:

$$\varepsilon(x) = \frac{\partial u}{\partial x} + z \frac{\partial \psi}{\partial x} \tag{7}$$

$$\gamma(x) = \frac{\partial w}{\partial x} + \psi \tag{8}$$

For establishing the dynamic equations of the beam, resultant axial force, the bending moment and the shear force are determined as

$$N = \int_A \sigma_{xx} dA, \quad M = \int_A z \sigma_{xx} dA, \quad Q = \int_A \tau_{xz} dA \tag{9}$$

where σ_{xx} is the normal stress, τ_{xz} is the transverse shear stress and A is the cross sectional area of the beam. Using Eqns. (4-5) and Eqns. (7-8) and Eqn. (9) one obtains the nonlocal Timoshenko constitutive relations

$$M - (e_0 a)^2 \frac{\partial^2 M}{\partial x^2} = EI \frac{\partial \psi}{\partial x} \tag{10}$$

$$Q - (e_0 a)^2 \frac{\partial^2 Q}{\partial x^2} = GA K \left(\frac{\partial w}{\partial x} + \psi \right) \tag{11}$$

where K is the shear correction factor used to compensate for the error due to constant shear stress assumption. I represents the moment of area of the cross section. Note that the bending moment and shear force given in Eqns. (10) and (11) reduces to that of the local (classical) Timoshenko model when the nonlocal scale coefficient, $(e_0 a)$ is set to zero.

Now consider a beam element of length dx which is axially loaded. The dynamic equation for the beam element of nonlocal bending moment and shear force is written as:

$$\frac{\partial Q}{\partial x} dx - P \frac{\partial^2 w}{\partial x^2} dx - \rho A \frac{\partial^2 w}{\partial t^2} dx = 0 \tag{12}$$

$$Q dx - \frac{\partial M}{\partial x} dx + \rho I \frac{\partial^2 \psi}{\partial t^2} dx = 0 \tag{13}$$

Eliminating Q first from Eqns. (12-13) and using (10-11) would lead to explicit relation of nonlocal bending moment and nonlocal shear force

$$M = EI \frac{\partial \psi}{\partial x} + (e_0 a)^2 \left(P \frac{\partial^2 w}{\partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} + \rho I \frac{\partial^3 \psi}{\partial x \partial t^2} \right) \tag{14}$$

$$Q = KA G \left(\psi + \frac{\partial w}{\partial x} \right) + (e_0 a)^2 \left(P \frac{\partial^3 w}{\partial x^3} + \rho A \frac{\partial^3 w}{\partial x \partial t^2} \right) \tag{15}$$

Again substituting the Eqns (14) and (15) into Eqns (12) and (13) one obtains nonlocal Timoshenko beam model

$$\begin{aligned} KA G \frac{\partial}{\partial x} \left(\psi + \frac{\partial w}{\partial x} \right) + (e_0 a)^2 \left(P \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^4 w}{\partial x^2 \partial t^2} \right) \\ - P \frac{\partial^2 w}{\partial x^2} - \rho A \frac{\partial^2 w}{\partial t^2} = 0 \end{aligned} \tag{16}$$

$$EI \frac{\partial^2 \psi}{\partial x^2} - KA G \left(\psi + \frac{\partial w}{\partial x} \right) - \rho I \frac{\partial^2 \psi}{\partial t^2} + (e_0 a)^2 \rho I \frac{\partial^4 \psi}{\partial x^2 \partial t^2} = 0 \tag{17}$$

Eqns. (16) and (17) are the consistent basic equations of the nonlocal Timoshenko beam model based on the constitutive relations (4) and (5). For the convenience and generality, the above nonlocal equations are decoupled to single sixth-order differential equation. These equations will be useful for deriving proper DQ models. In the present formulation the rotary inertia terms are neglected.

$$EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} - \frac{mEI}{KAG} \frac{\partial^4 w}{\partial x^2 \partial t^2} + P \frac{\partial^2 w}{\partial x^2} - \frac{PEI}{KAG} \frac{\partial^4 w}{\partial x^4} + (e_0 a)^2 \left(-m \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{mEI}{KAG} \frac{\partial^6 w}{\partial x^4 \partial t^2} - P \frac{\partial^4 w}{\partial x^4} + \frac{PEI}{KAG} \frac{\partial^6 w}{\partial x^6} \right) = 0 \quad (18)$$

$$EI \frac{\partial^4 \Psi}{\partial x^4} + m \frac{\partial^2 \Psi}{\partial t^2} - \frac{mEI}{KAG} \frac{\partial^4 \Psi}{\partial x^2 \partial t^2} + P \frac{\partial^2 \Psi}{\partial x^2} - \frac{PEI}{KAG} \frac{\partial^4 \Psi}{\partial x^4} + (e_0 a)^2 \left(-m \frac{\partial^4 \Psi}{\partial x^2 \partial t^2} + \frac{mEI}{KAG} \frac{\partial^6 \Psi}{\partial x^4 \partial t^2} - P \frac{\partial^4 \Psi}{\partial x^4} + \frac{PEI}{KAG} \frac{\partial^6 \Psi}{\partial x^6} \right) = 0 \quad (19)$$

Assuming the solution of the above equations as

$$w(x, t) = W(x) e^{i\omega t}, \quad \Psi(x, t) = \Psi(x) e^{i\omega t} \quad (20)$$

Where W is the amplitude of deflection of beam, ω is the frequency and $i = \sqrt{-1}$.

Substitution the Eqn. (20) in Eqns. (18) and (19) yields the following set of equations:

$$EI \frac{d^4 W}{dx^4} + m \omega^2 W - \frac{mEI\omega^2}{KAG} \frac{d^2 W}{dx^2} + P \frac{d^2 W}{dx^2} - \frac{PEI}{KAG} \frac{d^4 W}{dx^4} + (e_0 a)^2 \left(-m \omega^2 \frac{d^2 W}{dx^2} + \frac{mEI\omega^2}{KAG} \frac{d^4 W}{dx^4} - P \frac{d^4 W}{dx^4} + \frac{PEI}{KAG} \frac{d^6 W}{dx^6} \right) = 0 \quad (21)$$

$$EI \frac{d^4 \Psi}{dx^4} + m \omega^2 \Psi - \frac{mEI\omega^2}{KAG} \frac{d^2 \Psi}{dx^2} + P \frac{d^2 \Psi}{dx^2} - \frac{PEI}{KAG} \frac{d^4 \Psi}{dx^4}$$

$$+ (e_0 a)^2 \left(-m \omega^2 \frac{d^2 \Psi}{dx^2} + \frac{mEI\omega^2}{KAG} \frac{d^4 \Psi}{dx^4} - P \frac{d^4 \Psi}{dx^4} + \frac{PEI}{KAG} \frac{d^6 \Psi}{dx^6} \right) = 0 \quad (22)$$

If the shear deformation effect is neglected, it leads to dynamic equation based on nonlocal Euler-Bernoulli beam theory [17].

$$EI \frac{d^4 W}{dx^4} + (e_0 a)^2 \left(-m \omega^2 \frac{d^2 W}{dx^2} - P \frac{d^4 W}{dx^4} \right) + P \frac{d^2 W}{dx^2} + m \omega^2 W = 0 \quad (23)$$

It should be noted that when the scale coefficient (nonlocal parameter) term $(e_0 a)^2$ is set equal to zero, the nonlocal differential equations reduces to local (classical) differential equations.

The sixth-order derivative term appears in Eqns. (21) and (22) due to application of axial compressive load. In the present case only Eqn. (21) will be used. As analytical solutions of Eqn. (21) are difficult to obtain, a differential quadrature (DQ) approach has been undertaken for the solution. The DQ approach may be easy and useful for analyzing more complex problems such as CNTs with higher order theories. The nonlocal differential quadrature (N-DQ) formulation is proposed hereinafter

Differential Quadrature and Solution Procedure

In the differential quadrature method (DQM), derivatives (appearing in partial differential equation) of a function with respect to a space variable at a given interpolation point is approximated as a weighted linear summation of function values at all chosen interpolation points.

$$\frac{d^n F}{dx^n} \Big|_{x=x_i} = \sum_{j=1}^N C_{ij}^{(n)} F(x_j) \quad (24)$$

where N represents the number of grid points. Thus, DQM transforms the governing differential equation into a set of equivalent simultaneous equations. This is done by replacing the derivative term with equivalent weighting coefficients. For example, the first order derivative is equivalent to a weighting coefficient matrix

$$\frac{d}{dx} \equiv [C]_x^{(1)} \quad (25)$$

In this manner, the original governing differential equation is transformed into a set of distinct simultaneous algebraic equations. The implementation of this DQM technique depends on how accurately the weighting coefficient matrix is computed and the interpolation points (grid points) are distributed in the domain. The weighting coefficients of first order C_{ij} are expressed as [25].

$$C_{ij}^{(1)} = \frac{M(x_i)}{(x_i - x_j) M(x)}; \quad i, j = 1, 2, \dots, N; \quad i \neq j \quad (26)$$

The term $M(x_i)$ is defined as

$$M(x_i) = \prod_{j=1, j \neq i}^N (x_i - x_j); \quad i \neq j \quad (27)$$

and when $i = j$

$$C_{ij}^{(1)} = C_{ii}^{(1)} = - \sum_{k=1, k \neq i}^N C_{ik}^{(1)}; \quad i = 1, 2, \dots, N; \quad i \neq k; \quad i = j; \quad (28)$$

To obtain the weighting coefficients for the second, third and fourth order derivatives, the matrix multiplication procedure [17] is implemented: Similarly, second, third, and fourth, fifth and sixth order partial derivative are expressed in a matrix form as

$$\begin{aligned} \frac{d^2}{dx^2} &\equiv [C]_x^{(2)} \equiv [C]_x^{(1)} [C]_x^{(1)} \\ \frac{d^3}{dx^3} &\equiv [C]_x^{(3)} \equiv [C]_x^{(1)} [C]_x^{(1)} [C]_x^{(1)} \\ \frac{d^4}{dx^4} &\equiv [C]_x^{(4)} \equiv [C]_x^{(1)} [C]_x^{(1)} [C]_x^{(1)} [C]_x^{(1)} \\ \frac{d^5}{dx^5} &\equiv [C]_x^{(5)} \equiv [C]_x^{(1)} [C]_x^{(1)} [C]_x^{(1)} [C]_x^{(1)} [C]_x^{(1)} \\ \frac{d^6}{dx^6} &\equiv [C]_x^{(6)} \equiv [C]_x^{(1)} [C]_x^{(1)} [C]_x^{(1)} [C]_x^{(1)} [C]_x^{(1)} [C]_x^{(1)} \end{aligned} \quad (29)$$

For convenience and generality, the following nondimensional variables are introduced in the present nonlocal analysis:

$$\begin{aligned} X &= \frac{x}{L}, \quad \Omega^2 = \frac{m \omega^2 L^4}{EI}, \quad \bar{P} = \frac{PL^2}{EI}, \\ S &= \frac{EI}{KAGL^2}, \quad \alpha = \frac{e_0 a}{L} \end{aligned} \quad (30)$$

where L represent the length of the beam in nanosize. Here the terms EI represent the bending rigidity of the beam.

Using Eqn. (30) into (21) we arrive at governing equation in nondimensional form:

$$\begin{aligned} \frac{d^4 W}{dX^4} + \Omega^2 W_i - \Omega^2 S^2 \frac{d^2 W}{dX^2} + \bar{P} \frac{d^2 W}{dX^2} - \bar{P} S^2 \frac{d^4 W}{dX^4} \\ + \alpha^2 \left(-\Omega^2 \frac{d^2 W}{dX^2} + \Omega^2 S^2 \frac{d^4 W}{dX^4} - \bar{P} \frac{d^4 W}{dX^4} + \bar{P} S^2 \frac{d^6 W}{dX^6} \right) = 0 \end{aligned} \quad (31)$$

Using Eqns. (31) and (24) the associated nonlocal differential quadrature (N-DQ) governing equations are expressed as:

DQ Nonlocal Timoshenko Model for Nano-scale Beams:

$$\begin{aligned} \sum_{j=1}^N C_{ij}^{(4)} W_j + \Omega^2 W_i - \Omega^2 S^2 \sum_{j=1}^N C_{ij}^{(2)} W_j + \bar{P} \sum_{j=1}^N C_{ij}^{(2)} W_j - \bar{P} S^2 \sum_{j=1}^N C_{ij}^{(4)} W_j \\ + \alpha^2 \left(-\Omega^2 \sum_{j=1}^N C_{ij}^{(2)} W_j + \Omega^2 S^2 \sum_{j=1}^N C_{ij}^{(4)} W_j - \bar{P} \sum_{j=1}^N C_{ij}^{(4)} W_j + \bar{P} S^2 \sum_{j=1}^N C_{ij}^{(6)} W_j \right) = 0 \end{aligned} \quad (32)$$

DQ Nonlocal Euler-Bernoulli Model for Nano-scale Beams:

$$\sum_{j=1}^N C_{ij}^{(4)} W_j + \Omega^2 W_i + \bar{P} \sum_{j=1}^N C_{ij}^{(2)} W_j + \alpha^2 \left(-\Omega^2 \sum_{j=1}^N C_{ij}^{(2)} W_j - \bar{P} \sum_{j=1}^N C_{ij}^{(4)} W_j \right) = 0 \quad (33)$$

Where the deflection matrix W is expressed as

$$W_i = [W_1 \ W_2 \ W_3 \ \dots \ W_{N-1} \ W_N]^T \quad (34)$$

The term α in the Eqns. (32) and (33) takes care of the small size effects for nano-scale beams.

Nonlocal Boundary Conditions

The simply supported boundary conditions for nano-size beams at the two ends ($x = 0, L$) are specified by

$$w = 0 \tag{35}$$

$$M = EI \frac{\partial \Psi}{\partial x} + (e_0 a)^2 \left(P \frac{\partial^2 w}{\partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} + \rho I \frac{\partial^3 \Psi}{\partial x \partial t^2} \right) \tag{36}$$

It should be noted that M is the nonlocal bending moment and not the classical bending moment. Using the assumptions that $\Psi = dW/dx$, and using Eqn. (30) we have

$$\left[1 + \alpha^2 \bar{P} + \alpha^2 \frac{\rho I L^2}{EI} \omega^2 \right] \frac{d^2 W}{dX^2} + \alpha^2 \Omega^2 W = 0 \tag{37}$$

However, it is interesting to note that for simply supported boundary condition, the boundary equations for the classical beam model and nonlocal beam models are same. This is in view of $W = 0$ at the boundaries. Consequently the nonlocal effects are neglected there (Ref. [17] and [25]). Thus the boundary conditions reduces to

$$W = \frac{d^2 W}{dX^2} = 0 \quad \text{at} \quad X = 0, 1 \tag{38}$$

These nonlocal boundary conditions can be incorporated within the formulation during the determination of weighting coefficients. The weighting coefficients are updated as $\bar{C}_1, \bar{C}_2, \bar{C}_3, \bar{C}_4, \bar{C}_5, \bar{C}_6$. Details of the procedure can be seen in Pradhan and Murmu [23]. This DQ approach is known as MWCM method. The present article thus brings out the simplicity and generality of employing DQ approach in the field of nonlocal continuum mechanics. Thus the updated governing DQ equations incorporating boundary conditions is expressed as:

Updated DQ Nonlocal Timoshenko Model

$$\sum_{j=2}^{N-1} C_{ij}^{(4)} W_j + \Omega^2 W_i - \Omega^2 S^2 \sum_{j=2}^{N-1} C_{ij}^{(2)} W_j + \bar{P} \sum_{j=2}^{N-1} C_{ij}^{(2)} W_j - \bar{P} S^2 \sum_{j=2}^{N-1} C_{ij}^{(4)} W_j$$

$$+ \alpha^2 \left(-\Omega^2 \sum_{j=2}^{N-1} C_{ij}^{(2)} W_j + \Omega^2 S^2 \sum_{j=2}^{N-1} C_{ij}^{(4)} W_j - \bar{P} \sum_{j=2}^{N-1} C_{ij}^{(4)} W_j + \bar{P} S^2 \sum_{j=2}^{N-1} C_{ij}^{(6)} W_j \right) = 0 \tag{39}$$

Updated DQ Nonlocal Euler-Bernoulli Model

$$\sum_{j=2}^{N-1} C_{ij}^{(4)} W_j + \Omega^2 W_i + \bar{P} \sum_{j=2}^{N-1} C_{ij}^{(2)} W_j + \alpha^2 \left(-\Omega^2 \sum_{j=2}^{N-1} C_{ij}^{(2)} W_j - \bar{P} \sum_{j=2}^{N-1} C_{ij}^{(4)} W_j \right) = 0 \tag{40}$$

It should be noted that a single governing DQ equation is sufficient to represent the behavior of nano-size beams. Immense use of DQM in classical mechanics for complex problems is reported in literature. The DQ formulation used over here will be useful while obtaining quick solutions for complex problems in nanomechanics. One example is that of dealing with short tapered nano-cantilevers [29] and also with higher-order theories. DQ can be thus useful tool for dealing problems in computational nanomechanics.

For obtaining the frequency and critical buckling loads, Eqns. (39) and (40) can be easily transformed into an Eigen-value problem (using matrices from Eqn. (29)).

Nonlocal Timoshenko beam model for frequency analysis ($\bar{P} = 0$):

$$[K] \{W\} = \Omega^2 \{W\} \tag{41}$$

Nonlocal Timoshenko beam model for buckling analysis ($\Omega = 0$):

$$[K]^* \{W\} = \bar{P} \{W\} \tag{42}$$

Where $[K]$ and $[K]^*$ are cumulative matrix for nonlocal vibration and buckling beam models. The size of K matrix is in the order of $(N-2) \times (N-2)$.

Here the displacement matrix is given as:

$$W_i = [W_2 \ W_3 \ W_4 \ \dots \ W_{N-2} \ W_{N-1}]^T \tag{43}$$

In the present analysis two types of grid points are taken. The grid points are obtained based on uniformly distributed points and Chebyshev-Gauss-Lobatto Points [24]:

$$(X_i)_{uniform} = \frac{(i-1)}{(N-1)}; \quad i = 1, 2, \dots, N \quad (44)$$

$$(X_i)_{Cheby-Gauss-Lobatto} = \frac{1}{2} \left(1 - \cos \frac{(i-1)}{(N-1)} \pi \right); \quad i = 1, 2, \dots, N \quad (45)$$

Results and Discussions

DQ Nonlocal Frequency Analysis

Nondimensional frequencies for different nonlocal parameters are determined by employing the DQ approach. The nonlocal parameter here is defined as (e_0a/L) . Here L denotes the length of the nanobeam (such as length of single-walled carbon nanotubes). We consider L as the external characteristic length scale. The importance of nonlocal parameter can be seen in the earlier section. Here it should be noted that the nonlocal parameter (e_0a/L) has been normalized. Even when the dimensionless frequencies are introduced (which are non-dimensionalised with respect to length, bending rigidity and mass), the scale effects are still in attendance within the present nonlocal theory in nondimensional form. These scale-effect observations will be illustrated soon.

Frequency parameter $\sqrt{\Omega}$ determined from the present DQ approach are compared with the nonlocal elasticity based frequency results of that of Lu et al. [14]. Nondimensional frequencies (eigen-values) for the first four different modes of vibrations ($m=1, 2, 3$ and 4) are determined in the present analysis. The present results are shown for the nonlocal Euler-Bernoulli beam model. Fig.2 shows the variation of $\sqrt{\Omega}$ with nonlocal parameter (e_0a/L) . Nonlocal parameter values of 0.0, 0.2, 0.4, 0.6, and 0.8 are assumed. Similar range of values was considered by Lu et al. [14]. The results are plotted for nonlocal pinned-pinned boundary conditions. Here it should be noted that the local boundary condition and nonlocal boundary conditions are same. The figure shows that with increase of nonlocal parameter there is reduction in value of natural frequency. This reduction of frequencies is attributed to the small-scale effects. The decreasing effect with nonlocal parameter is highly pronounced for higher modes. In addition, the present DQ results are found to be agreeing excellently with the analytical results of Lu et al. [14].

Effect of Grid Points on Nonlocal Frequency Solution

Two types of DQ grid points are tested for the frequency solution of the nano-size beams. The grid points are the uniform grid points given by Eqn. (44). Non-uniform grid points are represented Gauss-Chebyshev-Lobatto points given by Eqn. (45). Error percentage is defined as $100 \times (\text{present result-Reddy [15]}/\text{Reddy [15]})$. Fig.3 shows the variation of error percentage with parameter $\mu = (e_0a/L)^2$. Nonlocal parameter values μ 0.0, 0.01, 0.02, 0.03, 0.04, 0.05 and 0.06 are assumed here. The present results are shown for first modes of vibration ($m = 1$). Plots for both uniform and nonuniform grid points are shown. The figure shows that the error percentage is independent of nonlocal parameters. Both uniform and nonuniform grid points show reliable results for the analysis of fundamental frequency. However, for the second mode of vibration, the use of uniform grid points and nonlocal Euler-Bernoulli model yield errors in DQ result for low nonlocal parameter values. This trend is shown in Fig.4. The error percentage using uniform grid points gradually decreases till a nonlocal parameter value of 0.04. This implies that uniform grid points could be used after a certain nonlocal parameter value. Also it is clear that nonlocal second mode of frequency results with nonuniform grid points is independent of nonlocal parameter. This shows that nonuniform grid points are good option for nonlocal frequency analysis.

Effect of Boundary Conditions on Nonlocal Frequency solution

The variation of nondimensional frequency parameter $\sqrt{\Omega}$ with nonlocal parameter is dependent on the boundary conditions. Fig.5 shows the variation of frequency parameter for simply-supported and cantilever boundary conditions. It is clearly seen that the boundary condition effects are opposite in nature. Unlike the frequency variation for simply supported - simply supported boundary conditions, the variation for cantilever beam is opposite in nature. There is a gradual increase in nondimensional frequency with the increase of nonlocal parameter. Thus the effect of small size effect on the frequency response is good for clamped-free boundary condition as it increases the natural frequency. Similar behavior with clamped-free boundary conditions for nonlocal buckling analysis is also observed in Murmu and Pradhan [17].

DQ Frequency Solution Based on Nonlocal Timoshenko Beam Theory

Nondimensional frequencies Ω for different nonlocal parameters are determined by employing DQ approach and nonlocal Timoshenko beam theory (Eqn.39). The effect of shear deformation is included in the analysis. A computer code is developed based on Eqn.(39). Five different aspect ratios, L/h are taken into consideration, viz 10, 20, 50, 70 and 100. The term ' h ' denotes the height of the nano beam. Shear correction factor is assumed as 5/6 in the computation. Poissons ratio of 0.3 is considered in the analysis. The variation of nondimensional frequency with nonlocal parameter is shown in Fig.6. The results are plotted for nonlocal pinned-pinned boundary conditions. Here very small values of nonlocal parameter are taken. The nonlocal parameter is defined as $\mu = (e_0 a/L)^2$. Nonlocal parameter values μ 0.0, 0.01, 0.02, 0.03, 0.04 and 0.05 are assumed here. The figure shows that with increase of nonlocal parameter there is reduction in value of natural frequency. Also it is seen that there is a significant effect of aspect ratio on the nonlocal frequency results. Lower aspect ratio leads to comparatively lower natural frequency. However similar trend of reduction of nonlocal frequencies is seen with all aspect ratio considered here. Similar conclusions can be found in Reddy [15]. However unlike the linear frequency behaviour for first mode, a nonlinear behaviour is noticed for the second mode of vibration (Fig.7).

DQ Nonlocal Buckling Analysis

Present DQ nonlocal Timoshenko beam analysis is extended for stability analysis of nano-size beams. The beam is assumed to be axially compressed. Critical buckling load for various nonlocal parameters μ is determined using the DQ formulation. The associated DQ formulation is depicted in Eqn.(39). For this DQ formulation, Chebyshev-Gauss-Lobatto grid points (Eqn.45) are undertaken. A nonlocal pinned-pinned boundary condition is considered here. Shear correction factor K is assumed as 5/6 in the computation. To see the functionality of sixth-order nonlocal DQ model, a comparison of present results and that of literature is carried out. A plot of critical load \bar{P} versus nonlocal parameter $\mu = (e_0 a/L)^2$ results is shown in Fig.8. Nonlocal parameter values μ 0.0, 0.01, 0.02, 0.03, 0.04 and 0.05 are assumed here. From the figure it is clear that the present DQ results agree excellently with that of Reddy [15].

Effect of Number of Grid Points on Nonlocal Buckling Solution

To see the effects of grid points on the nonlocal critical buckling load results, a convergence study is carried out. Aspect ratios of 100 and 20 are taken for the analysis. Fig.9 shows the variation of critical buckling load results versus number of grid points. The nonlocal parameter is taken as 0.025. The figure shows that six number of grid points are adequate for obtaining convergent nonlocal buckling results. No significant difference in convergent behavior is noticed for different aspect ratios. Fig.10 shows the variation of critical buckling load results versus number of grid points for nonlocal parameter value of 0.045. The figure also shows that six number of grid points are adequate for obtaining convergent nonlocal buckling results. This interprets that nonlocal parameter and aspect ratio does not have much influence on the convergent behaviour in nonlocal stability analysis.

Effect of Aspect Ratio and Modes on Nonlocal Buckling Solution

Nondimensional critical buckling load with nonlocal parameter μ is shown in Fig.11. The results are plotted for nonlocal pinned-pinned boundary conditions. Five different values of aspect ratios are assumed. They are 100, 70, 50, 20 and 10. Shear correction factor K is assumed as 5/6 in the computation. The figure shows that with increase of nonlocal parameter μ there is reduction in value of nondimensional critical buckling load. Also it is seen that there is a significant effect of aspect ratio on the nonlocal critical buckling load. Lower aspect ratio leads to comparatively lower nondimensional critical buckling load. However similar trend of reduction of nonlocal nondimensional critical buckling load is seen with all aspect ratio considered here.

Nondimensional critical buckling load with nonlocal parameter for the second mode is shown in Fig.12. Similar observation of behavior as that of first mode is also seen in this case. However unlike the linear buckling behaviour for first mode, a nonlinear behaviour is noticed for the second mode. This shows that the buckling load considering nonlocal Timoshenko theory and second mode are more influenced by nonlocal effects. In addition it is also observed that the effect of aspect ratio on nonlocal buckling load for second mode slightly reduces at higher nonlocal parameter values (Fig.12). However, this behaviour is unnoticeable for critical buckling load results. This

clearly reveals the importance of nonlocal theory in mechanical analysis of nano-size structures.

Finally, a comparison of variation of nonlocal critical buckling load with nonlocal parameter μ is done. Fig.13 shows the nonlocal curves using nonlocal Euler Bernoulli theory (NL-EBT) and nonlocal Timoshenko theory (NL-TBT). As found in classical theory, the nonlocal buckling load results with Timoshenko beam theory (NL-TBT) are always smaller than nonlocal Euler Bernoulli theory (NL-EBT). This shows the importance of using (NL-TBT) for short and stubby nano-tubes or nano-beams.

Application of Nonlocal-Timoshenko-Beam-Theory to Single-Walled Carbon Nanotubes (SWCNT)

Carbon nanotubes (CNT) are a new form of carbon discovered by Iijima [30]. It is configurationally equivalent to two dimensional graphene sheet rolled into a tube. Carbon nanotubes are the strongest and stiffest materials yet discovered in terms of tensile strength and elastic modulus respectively. They are unique nanostructures with remarkable electronic and mechanical properties. The CNT can be found in aerospace application as CNT based composites imparting high strength and low weight. They can also be potential material for nano-electronics in aerospace application.

In the present section we show the use of Nonlocal-Timoshenko-Beam-Theory in the vibration and buckling study of single-walled CNT (SWCNT). The effective properties of SWCNT are taken as: Youngs modulus $E = 1000$ GPa, mass density $\rho = 2300$ kg/m³, Poissons ratio $\nu = 0.19$ and the shear correction factor $\beta = 0.877$ is taken. The effects of scale coefficients (nonlocal parameter) are also illustrated in the Fig.14a-b. The scale coefficients or the nonlocal parameter were taken as $e_0a = 0.0$ nm, 0.5 nm, 0.1nm, 1.5nm and 2.0 nm. Note that here the dimensional value of nonlocal parameter is taken. These values were adopted because e_0a should be smaller than 2.0 nm for carbon nanotubes as described by Wang and Wang [31]. Note that the results associated with $e_0a = 0.0$ nm correspond to those of the local Timoshenko beam theory where the small-scale effect is ignored. Most single-walled nanotubes (SWNT) have a diameter of close to onenanometer, with a tube length that can be many thousands of times longer. The diameter of the SWCNT is assumed as 1.0 nm. And the length of the SWCNT is assumed to be in the range of 15 nm to 40 nanometers.

The nonlocal models in the Fig.14. takes into account the small scale effects. From this figure it is observed that very high and small values of frequency and buckling load are obtained for SWCNT. The frequencies are in the order of GHz and the buckling load is in the order of nano Newton. It is seen that with increase of length of CNT the frequencies and the buckling load decreases. However the nano-level effects are not seen with local model ($e_0a = 0$) (which neglects the scale effects). For nonlocal model the frequencies and the buckling load are found to be smaller. The difference for local and nonlocal model is pronounced for small length. The reduction in frequency and buckling load values through nonlocal model is accounted for the small scale effects which fail in traditional classical theory. Thus Nonlocal-Timoshenko-Beam-Theory can be important from application standpoint of nanostructures and aerospace sciences.

Conclusions

Presented herein is a differential quadrature analysis of nano size beams based on nonlocal elasticity and Timoshenko beam theory. The nonlocal theory takes into account the small scale effects in the analysis. A simplified uncoupled sixth-order governing differential equation for frequency and stability analysis of small-scale Timoshenko beams is derived using the nonlocal elasticity. The influence of in-plane loads on natural frequencies is included in the formulation. DQ based frequency and buckling results for various scale-based nonlocal parameters are shown to be agreeing excellently with existing nonlocal results. Both uniform and nonuniform grid points show reliable results for the fundamental frequency. It is seen that there is a significant effect of aspect ratio on the nonlocal frequency and buckling results. However unlike the linear buckling behaviour for first mode, a nonlinear behaviour is noticed for the second mode. The use of Nonlocal-Timoshenko-Beam-Theory in the vibration and buckling study of single walled CNT is shown. It is seen that with increase of length of CNT the frequencies and the buckling load decreases. However the nano-level effects are not seen with local model ($e_0a = 0$). For nonlocal model the frequencies and the buckling load are found to be smaller. The present study could open up a new approach of solution technique for the analysis of nano scale structures. Thus the Nonlocal-Timoshenko-Beam-Theory can be an important theory for analysis of nanostructures in aerospace sciences and technologies.

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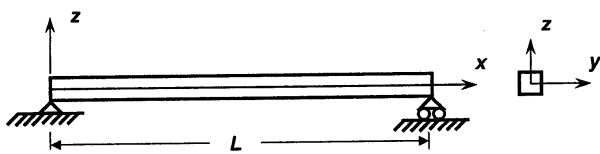


Fig.1 Coordinate System Assumed for the Nonlocal Timoshenko Beam Model

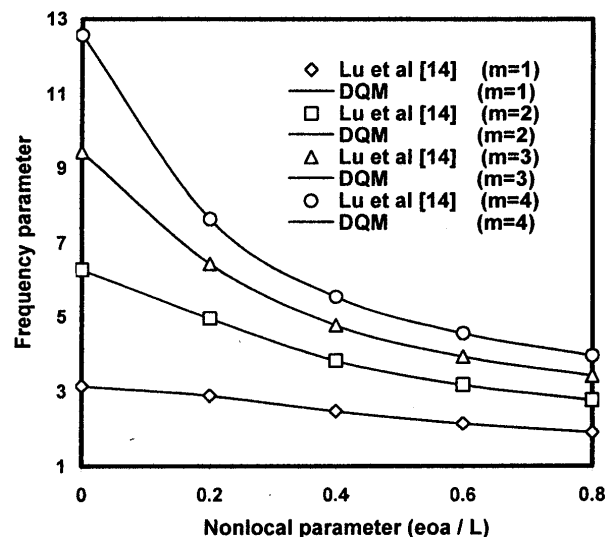


Fig.2 Comparison of Fundamental Frequency Results Obtained by DQM with that Available in the Literature. The Results are Shwon for all the Four Modes

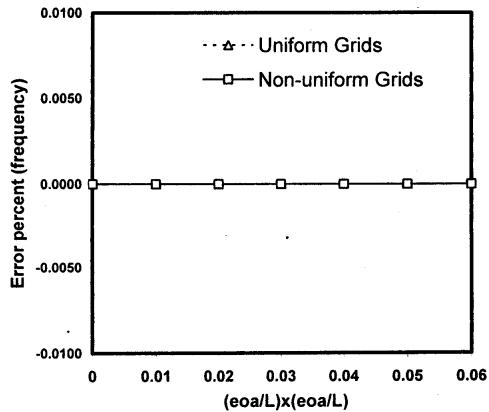


Fig.3 Comparison of DQ Frequency Results for Uniform and Nonuniform Grid Points. (First Mode of Vibration)

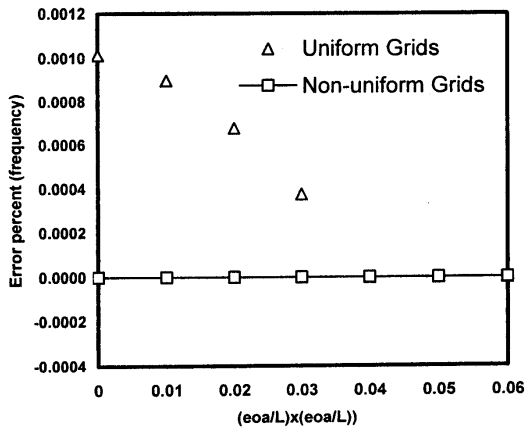


Fig.4 Comparison of DQ Frequency Results for Uniform and Nonuniform Grid Points. (Second Mode of Vibration)

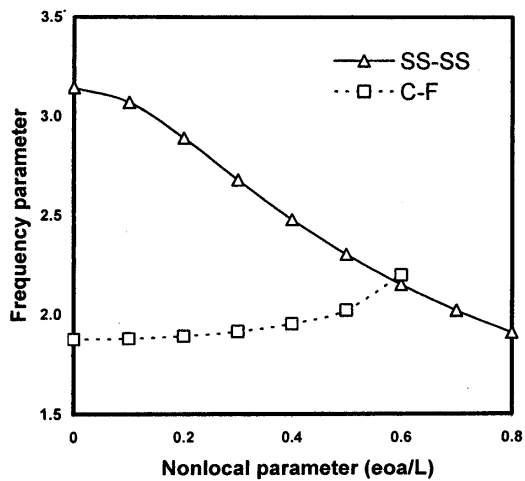


Fig.5 Comparison of Nonlocal Fundamental Frequency for Simply-Supported and Clamped-Free Boundary Conditions

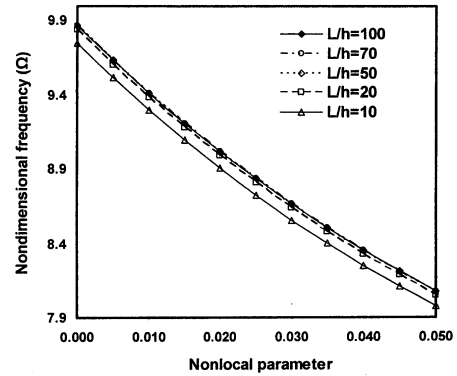


Fig.6 Variation of Fundamental Frequency with Nonlocal Parameter μ for Various Aspect Ratios with Nonlocal Timoshenko Model

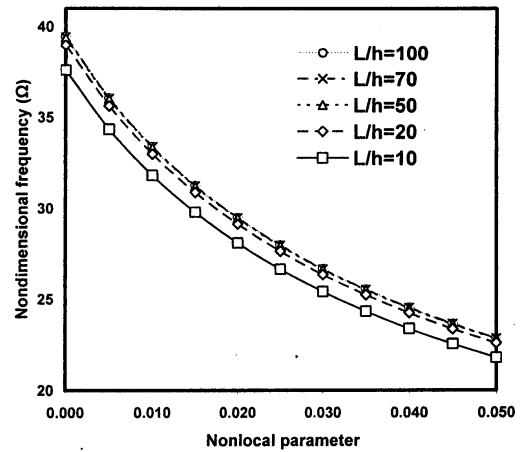


Fig.7 Variation of Frequency (Second Mode) with Nonlocal Parameter μ for Various Aspect Ratios with Nonlocal Timoshenko Model

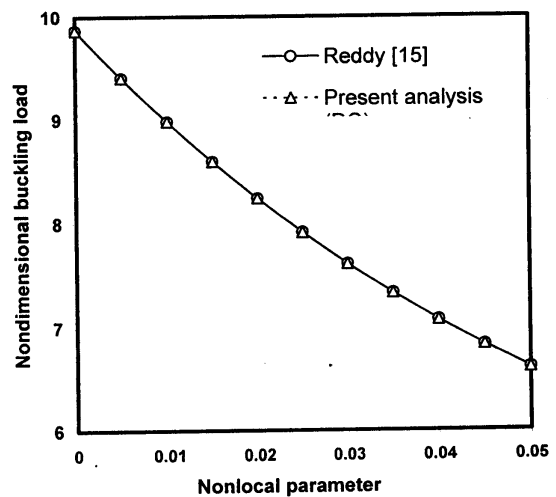


Fig.8 Comparison of DQ Buckling Results for with that Available in the Literature (Reddy [15])

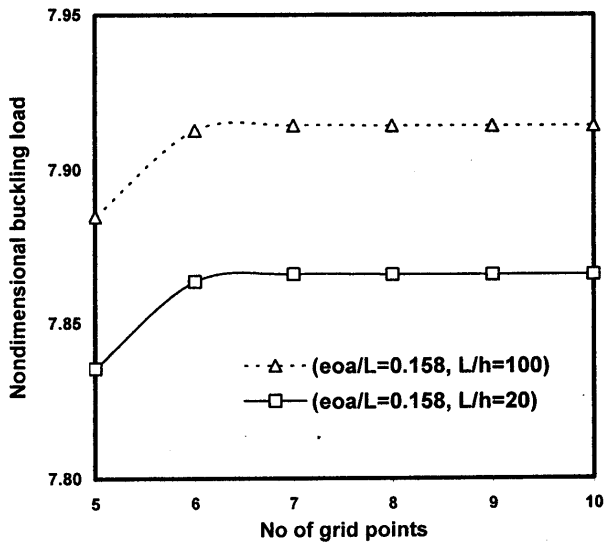


Fig.9 Variation of Nonlocal Buckling Loads with Number of DQ Grid Points for Different Aspect Ratios and Nonlocal Parameter as 0.025

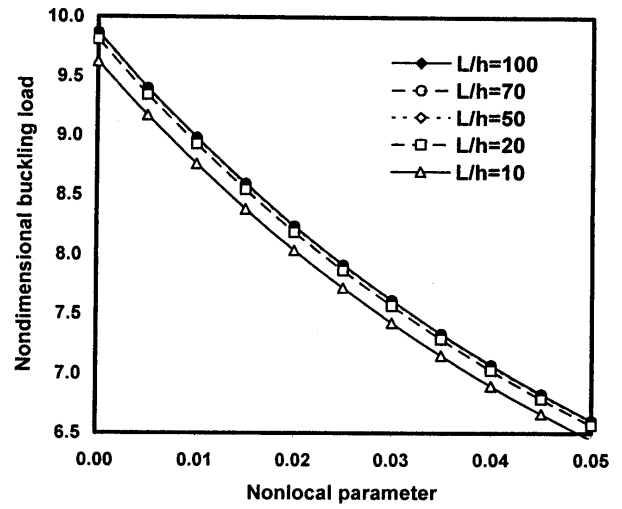


Fig.11 Variation of Critical Buckling Load with Nonlocal Parameter for Various Aspect Ratios with Nonlocal Timoshenko Model

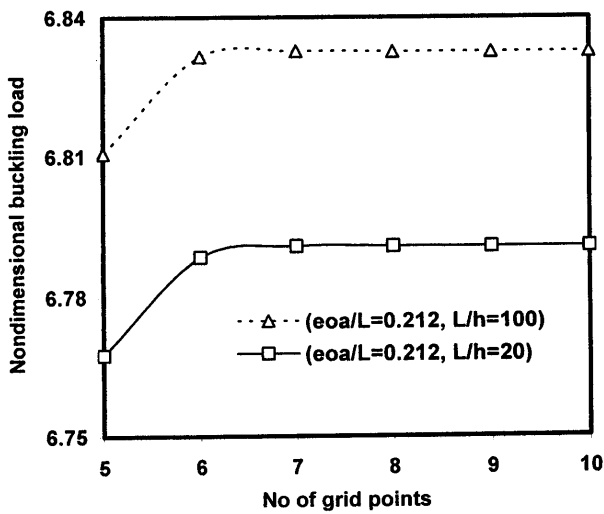


Fig.10 Variation of Nonlocal Buckling Loads with Number of DQ Grid Points for Different Aspect Ratios and Nonlocal Parameter as 0.045

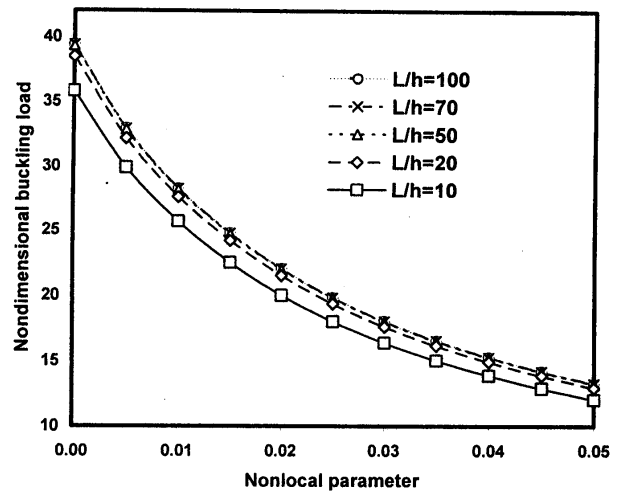


Fig.12 Variation of Buckling Load (Second Mode) with Nonlocal Parameter for Various Aspect Ratios with Nonlocal Timoshenko Model

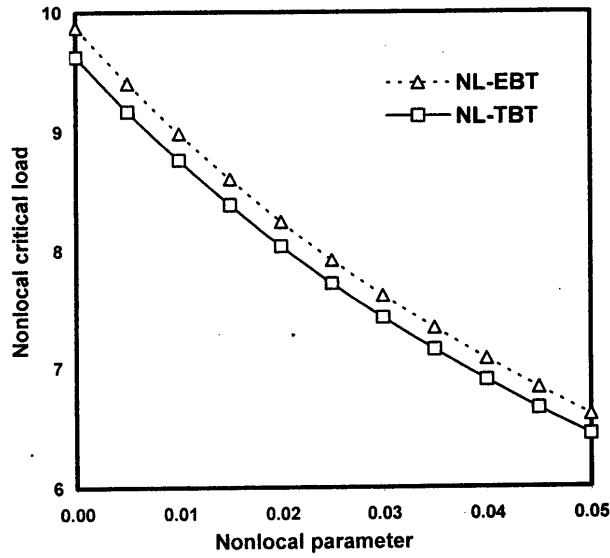


Fig.13 Comparison of Variation of Critical Load with Nonlocal Parameter for Nonlocal Euler-Bernoulli Theory and Nonlocal Timoshenko Beam Theory

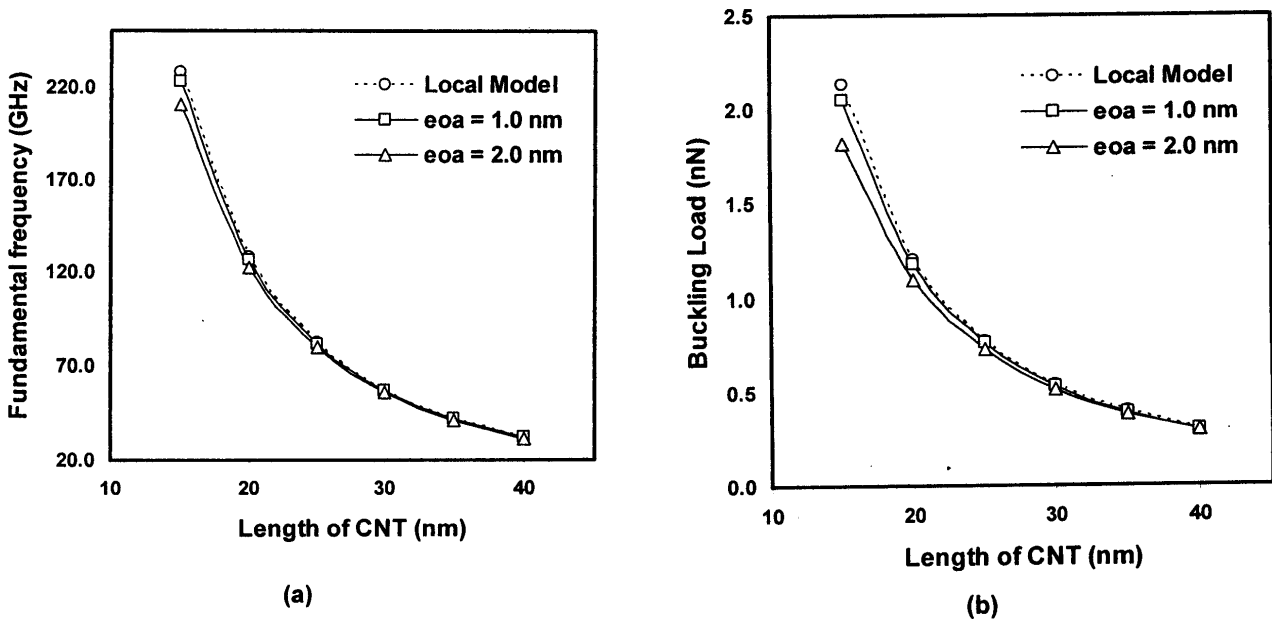


Fig.14 Variation of (a) Fundamental Frequency and (b) Buckling Load of CNT with the Length of CNT Computed Using Traditional Local and Nonlocal Models