

NON-LOCAL ANALYSES OF TAPERED BEAMS

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Abstract

In the present article bending, vibration and buckling analyses of a tapered beam using Eringen non-local elasticity theory is being carried out. The associated governing differential equations are solved employing Rayleigh-Ritz method. Both Euler-Bernoulli and Timoshenko beam theories are considered in the analyses. Present results are in good agreement with those reported in literature. Non-local analyses for tapered beam with simply supported - simply supported (SS), clamped - simply supported (CS) and clamped - free (CF) boundary conditions are conducted and discussed. It is observed that the maximum deflection increases with increase in non-local parameter value for SS and CS boundary conditions. Further, vibration frequency and critical buckling load decrease with increase in non-local parameter value for SS and CS boundary conditions. Non-local parameter effect on deflection, frequency and buckling load for CF supports is found to be opposite in nature to that of SS and CS supports. In case of thick beams non-local structural response is observed to be sensitive to length to thickness ratio.

Keywords: non local theory, Rayleigh-Ritz method, tapered beam, bending, buckling, vibration, and boundary conditions

Nomenclature

a_i	= arbitrary constant	ω	= natural frequency
$A, A(x)$	= areas of cross section	Π_P	= total potential energy
b	= width of beam	ρ	= density of material
E	= modulus of elasticity	σ_{xx}	= bending stress
G	= modulus of rigidity	ϵ_{xx}	= bending strain
h, h_0, h_1	= heights of beam	τ_{xz}	= shear stress
$I, I(x)$	= moments of inertia	ϵ_{xz}	= shear strain
k	= shape factor	$\varphi(x)$	= rotation due to shear
L	= length of beam	μ	= nonlocal factor
M	= bending moment	m_0, m_2	= mass inertias
n_i	= integer value		
N	= axial load		
P	= concentrated load		
P_{cr}	= critical buckling load		
T_{max}	= maximum kinetic energy		
u_i	= displacement		
U	= strain energy due to bending		
V_E	= work done by external force		
V_S	= potential energy due to shear		
w	= transverse deflection of beam		

Introduction

Most classical continuum theories are based on hyper elastic constitutive relations which assume that the stress at a point is a function of strain at that point. On the other hand, the non-local continuum mechanics assumes that the stress at a point is a function of strains at all points in the continuum. Such theories contain information about the forces between atoms, and the internal length scale is introduced into the constitutive equations as a material parameter. The non-local theory of elasticity has been used

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to study lattice dispersion of elastic waves, wave propagation in composites, dislocation mechanics, fracture mechanics, surface tension on fluids, etc. Work on nonlocal elasticity is introduced by Eringen [1-3] and Eringen and Edelen [4]. Peddieson et. al [5] employed nonlocal continuum model in nanotechnology. Pin et. al [6] employed a nonlocal beam model in carbon nanotubes. Reddy [7] applied nonlocal theories and reported bending, buckling and vibration results of beams. Wang et. al [8] applied nonlocal continuum mechanics and conducted static analysis of micro- and nano-structures. Heireche et. al. [9] employed nonlocal elasticity and studied sound wave propagation in single-walled carbon nanotubes. Zhou [10] studied free vibration of tapered beams. Maalek [11] investigated shear deflections of tapered timoshenko beams. Ece et. al [12] reported vibration response of variable cross section beam. Ganesan and Zabihollah [13-14] conducted parametric study on vibration of tapered beams. They employed higher order finite element analysis. Reddy [15] found relation between bending solutions of classical and shear deformation beam theories. Reddy and Wang [16] also discussed relationship between solutions of the classical and shear deformation plate theories. Liew et. al [17] studied free vibration and buckling of shear-deformable plates based on meshfree method. Brown and Stone [18] employed Rayleigh-Ritz method and studied composite materials. Leissa [19] reported historical bases of Rayleigh-Ritz method. Shames and Dym [20] reported energy and finite element methods for various structural mechanics problems. Reddy and Pang [21] reported non local continuum theories for beams.

The tapered beams are increasingly being used in engineering applications, such as turbine blades, helicopter blades and yokes, robot arms and satellite antennas. Here stiffness of the structure is varied along the length of the beam. Nonlocal analysis of tapered beams is important and little information is available in the literature. Thus in the present work authors have attempted to carry out nonlocal analyses of tapered beams with various boundary conditions. This work includes bending, buckling and vibration of the beams.

Formulation

Nonlocal Theory

The stress field at a point x in an elastic continuum depends on the strain field at the point (hyper elastic case) as well as strains at all other points of the body. Eringen [3] attributed this fact to the atomic theory of lattice dynamics and experimental observations on phonon dis-

persion. Thus, the non-local stress tensor σ at point x is expressed as an integral form over the body

$$\sigma = \int_V K(|x' - x|, \tau) t(x') dv(x') \quad (1)$$

where t is the classical, macroscopic stress tensor at point x' in the body and the nonlocal kernel function $K(|x' - x|, \tau)$ which brings the influence of strain at distant points x' to the stress at x . ($|x' - x|$) is the distance in Euclidean norm. τ is a material constant that depends on internal and external characteristic lengths such as the lattice spacing and wavelength, respectively. The macroscopic stress ' t ' at a point x in a Hookean solid is related to the strain ε at the point by the generalized Hook's law

$$t(x) = C(x) : \varepsilon(x) \quad (2)$$

where C is the fourth-order elasticity tensor and $:$ denotes the double-dot product. The constitutive Eqs. (1) and (2) together define the non-local constitutive behaviour of a Hookean solid [7]. Further, Eqn.(1) represents the weighted average of the contributions of the strain field of all points in the body to the stress field at a point. This represents the integral constitutive relations in an equivalent differential form as

$$(1 - \tau^2 l^2 \nabla^2) \sigma = t, \tau = \frac{e_0 a}{l} \quad (3)$$

where e_0 is a material constant. a and l are the internal and external characteristic lengths, respectively. Using Eqs. (2) and (3), stress resultants are expressed in terms of the strains in different beam theories. In the local theory the relation of stress resultants and strains are represented as linear algebraic equations. While in non-local theory the relation of stress resultants and strains are represented as differential equations. For homogeneous isotropic beams the non-local behavior is assumed to be negligible in the thickness direction. The constitutive relation for macroscopic stress take the special relation for beams

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E \varepsilon_{xx}, \sigma_{xz} - \mu \frac{\partial^2 \sigma_{xz}}{\partial x^2} = 2G \varepsilon_{xz} \quad (4)$$

where $\mu = e_0^2 a^2$.

The axial force-strain relation is given by

$$N - \mu \frac{\partial^2 N}{\partial x^2} = E \varepsilon_{xx}^0 \quad (5)$$

where $N = \int_A \sigma_{xx} dA$. The x -axis is considered along the geometric centroid of the beam. In Euler-Bernoulli beam theory, the constitutive relation is given by

$$M^E - \mu \frac{\partial^2 M^E}{\partial x^2} = E I \kappa^E \quad (6)$$

where $\kappa^E = -\frac{\partial^2 w^E}{\partial x^2}$ and $M^E = \int_A z \sigma_{xx} dA$.

The superscript ‘E’ denotes the quantities associated with Euler-Bernoulli beam theory. In case of the Timoshenko beam theory we have additional M^T and Q^T terms. The constitutive relation is given as

$$M^T - \mu \frac{\partial^2 M^T}{\partial x^2} = E I \kappa^T, \quad Q^T - \mu \frac{\partial^2 Q^T}{\partial x^2} = G A K_s \gamma^T \quad (7)$$

where $Q = \int_A \sigma_{xz} dA$. K_s denotes the shear correction factor. $\kappa^T = \frac{\partial \phi^T}{\partial x}$ and $\gamma^T = \frac{dw^T}{dx} + \phi^T$.

The superscript ‘T’ denotes the quantities associated with the Timoshenko beam theory [7].

In the present work deflections, natural frequencies and critical loads for uniform and tapered beams (Fig.1) with various boundary condition are calculated. The moment of inertia and cross section area are changing along the beam axis.

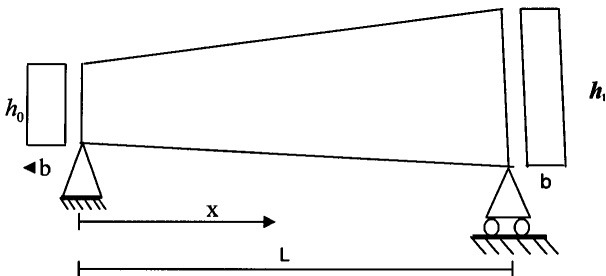


Fig.1 A Schematic Diagram of Tapered Beam with Simply Supported - Simply Supported Boundary Conditions

The moment equation for Euler-Bernoulli non-uniform beam theory is written as

$$M^E = -E I(x) \frac{\partial^2 w}{\partial x^2} + \mu \left[\frac{\partial}{\partial x} \left(N^E \frac{\partial w}{\partial x} \right) - q + m_0 \frac{\partial^2 w}{\partial t^2} - m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} \right] \quad (8)$$

while moment equation Timoshenko non-uniform beam theory is expressed as

$$M^T = E I(x) \frac{\partial \phi}{\partial x} + \mu \left[\frac{\partial}{\partial x} \left(N^T \frac{\partial w}{\partial x} \right) - q + m_0 \frac{\partial^2 w}{\partial t^2} - m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} \right] \quad (9)$$

Shear force is written as

$$Q^T = G A(x) K_s \left(\phi + \frac{\partial w}{\partial x} \right) + \mu \frac{\partial}{\partial x} \left[-q + \frac{\partial}{\partial x} \left(N \frac{\partial w}{\partial x} \right) + m_0 \frac{\partial^2 w}{\partial t^2} \right] \quad (10)$$

Bending

Flexural response of the beams are computed by employing Rayleigh-Ritz method. Strain energy for bending is expressed as

$$U = \frac{1}{2E} \int_0^L \frac{M^2}{I(x)} dx \quad (11)$$

Beam flexural equation is written as

$$M = -E I(x) \frac{d^2 w}{dx^2} \quad (12)$$

Putting (12) in (11)

$$U = \frac{1}{2} \int_0^L E I(x) \left(\frac{d^2 w}{dx^2} \right)^2 dx \quad (13)$$

Strain energy due to shear force is written as

$$V_s = \frac{1}{2G} \int_0^L \frac{Q^2}{A(x)} dx \quad (14)$$

Work done by uniformly distributed load (UDL) is expressed as

$$V_E = - \int_0^L q w dx \quad (15)$$

Total potential energy for the Euler-Bernoulli beam with UDL is expressed as

$$\Pi_p = U + V_E \tag{16}$$

where U is strain energy due to bending. V_E is the work done by external force. For purely bending analysis $N^E = 0$, $m_0 = 0$ and $m_2 = 0$ are incorporated in Eqn (8). Equation (8) is rewritten as

$$M^E = -E I(x) \frac{\partial^2 w}{\partial x^2} + \mu (-q) \tag{17}$$

Strain energy for bending

$$U = \frac{1}{2E} \int_0^L \frac{(M^E)^2}{I(x)} dx \tag{18}$$

Work done by UDL is expressed as

$$V_E = - \int_0^L q w dx \tag{19}$$

In Rayleigh-Ritz Method the component of approximate displacement w is approximated as functions containing a finite number of independent parameters, These parameters are determined such that the total potential energy computed on the basis of the approximate displacements is a minimum.

For a given structural system, w is assumed as

$$w = a_1 u_1(x) + a_2 u_2(x) + \dots + a_n u_n(x) \tag{20}$$

where a_1, a_2, \dots, a_n are the linear independent parameters and u_1, u_2, \dots, u_n are the continuous functions of the co-ordinate x . u_1, u_2, \dots, u_n satisfy all the kinematics boundary conditions for all values of the constant a_1, a_2, \dots, a_n . The total potential energy is a function of a_1, a_2, \dots, a_n .

When the system is in equilibrium,

$$\sum_{i=1}^n \frac{\partial \Pi}{\partial a_i} = 0 \tag{21}$$

Eqn. (21) is satisfied only if

$$\frac{\partial \Pi}{\partial a_1} = 0 \quad \frac{\partial \Pi}{\partial a_2} = 0 \quad \dots \dots \frac{\partial \Pi}{\partial a_n} = 0 \tag{22}$$

From Eqn. (22) a_1, a_2, \dots, a_n are determined. Incorporating a_1, a_2, \dots, a_n in Eqn.(20) approximate displacement is determined.

Similarly beam bending deflections are computed by employing Rayleigh-Ritz method for the Timoshenko beam. Total potential energy for Timoshenko beam with UDL is expressed as

$$\Pi_p = U + V_S + V_E \tag{23}$$

For bending analysis of Timoshenko beam $N^T = 0$, $m_0 = 0$ and $m_2 = 0$ are put in Eqn. (9) and we get

$$M^T = -E I(x) \frac{\partial \phi}{\partial x} + \mu (-q) \tag{24}$$

Strain energy for bending is

$$U = \frac{1}{2E} \int_0^L \frac{(M^T)^2}{I(x)} dx \tag{25}$$

For bending analysis $N = 0$ and $m_0 = 0$. q is independent of x . Putting these values in Eqn. (10) we get

$$Q^T = GA(x) K_s \left(\phi + \frac{\partial w}{\partial x} \right) \tag{26}$$

Strain energy due to shear force is

$$V_S = \frac{1}{2G} \int_0^L \frac{(Q^T)^2}{A(x)} dx \tag{27}$$

work done by UDL is written as

$$V_E = - \int_0^L q w dx \tag{28}$$

In Timoshenko beam approximate displacement w and rotation ϕ are functions containing a finite number of independent parameters. These parameters are determined so that the total potential energy computed on the basis of

the approximate displacements is a minimum. w and ϕ are expressed as

$$w = a_1 u_1(x) + a_2 u_2(x) + \dots + a_n u_n(x)$$

$$\phi = b_1 v_1(x) + b_2 v_2(x) + \dots + b_n v_n(x) \quad (29)$$

where a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n are linear independent parameters and u_1, u_2, \dots, u_n and v_1, v_2, \dots, v_n are the continuous functions of the co-ordinate x . All the kinematics boundary conditions for all value of the constant a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n are satisfied. The total potential energy is a function of a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n . System is in equilibrium implies

$$\sum_{i=1}^n \frac{\partial \Pi}{\partial a_i} \delta a_i = 0 \text{ and } \sum_{i=1}^n \frac{\partial \Pi}{\partial b_i} \delta b_i = 0 \quad (30)$$

Eqn.(30) is satisfied for arbitrary values of $\delta a_i, \delta b_i$. Thus

$$\frac{\partial \Pi}{\partial a_1} = 0 \quad \frac{\partial \Pi}{\partial a_2} = 0 \quad \dots \quad \frac{\partial \Pi}{\partial a_n} = 0$$

$$\frac{\partial \Pi}{\partial b_1} = 0 \quad \frac{\partial \Pi}{\partial b_2} = 0 \quad \dots \quad \frac{\partial \Pi}{\partial b_n} = 0 \quad (31)$$

From Eqn.(31) a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n are determined and putting these values in Eqn.(29) we get approximate displacement w and rotation ϕ .

Vibration

Vibration frequencies of the beams are computed by employing Rayleigh-Ritz method. Total potential energy for Euler-Bernoulli beam is expressed as

$$\Pi_P = V_{MAX} - T_{MAX} \quad (32)$$

where V_{MAX} and T_{MAX} are total maximum strain energy due to bending and maximum kinetic energy, respectively. Kinetic energy is written as

$$T = \frac{1}{2} \int_0^l \left(\frac{dw(x,t)}{dt} \right)^2 dm = \frac{1}{2} \int_0^l \left(\frac{dw(x,t)}{dt} \right)^2 \rho A(x) dx \quad (33)$$

where $dm = \rho A(x)$. The maximum kinetic energy can be obtained by assuming a harmonic variation $w(x, t) = w(x) \cos \omega t$. Maximum kinetic energy is expressed as

$$T_{max} = \frac{\omega^2}{2} \int_0^L \rho A(x) w^2(x) dx \quad (34)$$

For vibration analysis in Eqn. (8) $q = 0$ and $N = 0$ are considered and Eqn. (8) is rewritten as

$$M^E = -EI(x) \frac{\partial^2 w}{\partial x^2} + \mu \left[m_0 \frac{\partial^2 w}{\partial t^2} \right] \quad (35)$$

Maximum value of potential energy is expressed as

$$V_{max} = \frac{1}{2E} \int_0^L \frac{(M^E)^2}{I(x)} dx \quad (36)$$

In vibration analysis displacement w is expressed as an approximate function of independent parameters satisfying kinematic boundary conditions. For maximum total potential energy, we have

$$\frac{\partial \Pi}{\partial a_i} = 0 \quad (37)$$

After simplifying Eqn. (37) we get

$$\sum_{i=1}^n (A_i - \omega^2 D_i) a_i = 0 \quad (38)$$

where

$$A_i = \int_0^L EI(x) \left(\frac{d^2 w}{dx^2} \right) dx \text{ and } D_i = \int_0^L \rho A(x) w dx$$

We have a homogenous system of n number of equations. For a nontrivial solution the determinant of the coefficients is equal to zero. Thus we get

$$|A_i - \omega^2 D_i| = 0 \quad (39)$$

From Eqn. (39) frequency ω is determined. In a similar way frequency for Timoshenko beam is computed. Total potential energy for Timoshenko beam is expressed as

$$\Pi_P = V_{MAX} + V_S - T_{MAX} \quad (40)$$

V_{MAX} , V_S and T_{MAX} represent maximum strain energy due to bending, strain energy due to shear and maximum kinetic energy, respectively. Maximum kinetic energy T_{MAX} is expressed as in Eqn. (34). For vibration analysis $q = 0$ and $N = 0$ are incorporated in Eqn. (9). Equation (9) is rewritten as

$$M^T = -EI(x) \frac{\partial \phi}{\partial x} + \mu \left[m_0 \frac{\partial^2 w}{\partial t^2} + m_2 \frac{\partial^3 \phi}{\partial x \partial t^2} \right] \quad (41)$$

Maximum value of potential energy is

$$V_{MAX} = \frac{1}{2E} \int_0^L \frac{(M^T)^2}{I(x)} dx \quad (42)$$

For vibration analysis $q = 0$ and $N = 0$ are incorporated in Eqn.(10). Equation (10) is rewritten as

$$Q^T = GA(x) K_s \left(\phi + \frac{\partial w}{\partial x} \right) + \mu \left[m_0 \frac{\partial^2 w}{\partial t^2} \right] \quad (43)$$

strain energy due to shear force is written as

$$V_S = \frac{1}{2G} \int_0^L \frac{(Q^T)^2}{A(x)} dx \quad (44)$$

w and ϕ are approximate function with independent parameter that satisfy kinematic boundary conditions. For maximum total potential energy

$$\frac{\partial \Pi}{\partial a_i} = 0 \quad \text{and} \quad \frac{\partial \Pi}{\partial b_i} = 0 \quad (45)$$

After simplifying Eqn. (45) we get

$$\sum_{i=1}^n (A_i - \omega^2 D_i) a_i = 0 \quad (46)$$

where

$$A_i = \int_0^L EI(x) \left(\frac{d^2 w}{dx^2} \right)^2 dx \quad \text{and} \quad D_i = \int_0^L (\rho A(x) w + \rho I \phi) dx$$

We have a homogenous system of n number of eqns. For a nontrivial solution the determinant of the coefficients is equal to zero. Thus we get

$$|A_i - \omega^2 D_i| = 0 \quad (47)$$

From Eqn. (47) frequency ω is determined.

Buckling

Critical buckling loads are computed by employing Rayleigh-Ritz method. Total potential energy of the column with Euler-Bernoulli beam theory is expressed as

$$\Pi P = V_{MAX} + V_P \quad (48)$$

where V_{MAX} is maximum strain energy for bending and V_P is work done due to external load.

For buckling analysis $q = 0$, $m_0 = 0$, $m_2 = 0$ are put in Eqn. (8). Equation (8) is rewritten as

$$M^E = -EI(x) \frac{\partial^2 w}{\partial x^2} + \mu \left[\frac{\partial}{\partial x} \left(N^E \frac{\partial w}{\partial x} \right) \right] \quad (49)$$

Maximum strain energy for bending is expressed as

$$V_{\max} = \frac{1}{2E} \int_0^L \frac{(M^E)^2}{I(x)} dx \quad (50)$$

Work done due to external load is written as

$$V_P = -\frac{P}{2} \int_0^L \left(\frac{dw}{dx} \right)^2 dx \quad (51)$$

In buckling analysis w is approximated with independent parameters satisfying kinematic boundary conditions. For maximum total potential energy

$$\frac{\partial \Pi}{\partial a_i} = 0 \quad (52)$$

After simplifying Eqn. (52) we get

$$\sum_{i=1}^n (A_i - P D_i) a_i = 0 \quad (53)$$

where

$$A_i = \int_0^L EI(x) \left(\frac{d^2 w}{dx^2} \right)^2 dx \quad \text{and} \quad D_i = \int_0^L \frac{1}{2} P \left(\frac{dw}{dx} \right)^2 dx$$

We have a homogenous system of n number of eqns. For a nontrivial solution the determinant of the coefficients is equal to zero. Thus we get

$$|A_i - P D_i| = 0 \tag{54}$$

From Eqn. (54) buckling load P is determined. Total potential energy of the column with Timoshenko beam theory is expressed as

$$\Pi p = V_{MAX} + V_S + V_P \tag{55}$$

where V_{MAX} , V_S and V_P represent maximum strain energy for bending, strain energy due to shear force and work done by external load, respectively. For buckling analysis $q = 0$, $m_0 = 0$ and $m_2 = 0$ are incorporated in Eqn (9). Equation (9) is rewritten as

$$M^T = -EI(x) \frac{\partial \phi}{\partial x} + \mu \left[N \frac{\partial^2 w}{\partial t^2} \right] \tag{56}$$

Maximum strain energy for bending is written as

$$V_{MAX} = \frac{1}{2E} \int_0^L \frac{(M^T)^2}{I(x)} dx \tag{57}$$

For buckling analysis $q = 0$ and $m_0 = 0$ are incorporated in Eqn. (10). Equation (10) is rewritten as

$$Q = GA(x) K_s \left(\phi + \frac{\partial w}{\partial x} \right) + \mu \left[N \frac{\partial^3 w}{\partial x^3} \right] \tag{58}$$

Strain energy due to shear force is written as

$$V_S = \frac{1}{2G} \int_0^L \frac{(Q^T)^2}{A(x)} dx \tag{59}$$

Work done due to external load is expressed as

$$V_P = -\frac{P}{2} \int_0^L \left(\frac{dw}{dx} \right)^2 dx \tag{60}$$

w and ϕ are approximate functions with independent parameters satisfying kinematic boundary conditions. For maximum total potential energy

$$\frac{\partial \Pi}{\partial a_i} = 0 \quad \text{and} \quad \frac{\partial \Pi}{\partial b_i} = 0 \tag{61}$$

After simplification, we get

$$\sum_{i=1}^n (A_i - P D_i) a_i = 0 \tag{62}$$

where

$$A_i = \int_0^L EI(x) \left(\frac{d^2 w}{dx^2} \right) dx \quad \text{and} \quad D_i = \int_0^L \frac{1}{2} P \left(\frac{dw}{dx} \right) dx$$

Thus we have a homogenous system of n number of eqns. For a nontrivial solution the determinant of the coefficients is equal to zero. Thus we get

$$|A_i - P D_i| = 0 \tag{63}$$

From Eqn. (63) critical buckling load P is determined.

Results and Discussions

Bending of Beam

In the nonlocal flexural beam analysis following configurations are considered (Reddy [7]). Length of beam $L = 10.0m$, width $b = 1.0m$, height h of 0.1m, 1.0m and 2.0m,

Table-1 : Non-dimensional Maximum Deflection at Free End $\left[\hat{w} = w \left(\frac{EI}{qL^4} \right) \right]$ in Clamped - Free Uniform Beam

L/h	μ	Peddieson et.al [5]	Present Result		% of Difference
		EBT	EBT	TBT	
100	0.0	0.1250	0.1250	0.1250	0.000
	0.5	0.1225	0.1225	0.1225	0.000
	1.0	0.1200	0.1200	0.1200	0.000
	1.5	0.1175	0.1175	0.1175	0.000
	2.0	0.1150	0.1150	0.1150	0.000
	2.5	0.1125	0.1125	0.1125	0.000
	3.0	0.1100	0.1100	0.1100	0.000
	3.5	0.1075	0.1075	0.1075	0.000
	4.0	0.1050	0.1050	0.1050	0.000
	4.5	0.1025	0.1025	0.1025	0.000
5.0	0.1000	0.1000	0.1000	0.000	

Young’s modulus $E = 30 \times 10^6 \text{ N/m}^2$, Poisson ratio $\nu = 0.3$, UDL per unit length $q = 1 \text{ N/m}$, moment of inertia $I = \left(\frac{b \times h^3}{12}\right) m^4$, cross section of beam $A = (b \times h) m^2$, density $\rho = 1 \text{ kg/m}^3$ and shear correction factor $k = \left(\frac{5}{6}\right)$ are considered.

For tapered beam height h is assumed to be varying linearly along the beam length (Fig.1). Moment of inertia and cross-section area are expressed as $I_1 = I_0 (1 + (x/L)) m^4$ and $A_1 = A_0 (1 + (x/L)) m^2$, respectively. simply supported - simply supported (SS),

clamped - simply supported (CS) and clamped - free (CF) boundary conditions are considered in the analysis.

Maximum deflection for SS, CS and CF beams are computed as mentioned in Eqns. (20, 29). Employing Euler-Bernoulli theory (EBT) and Timoshenko beam theory (TBT) for CF and SS beams results are listed in Tables-1 to 2, respectively. From Table-1, one could observe that the present nonlocal results are exactly matching with those reported by Peddieson et. al [5]. Peddieson et. al [5]’s nonlocal work is limited to Euler-Bernoulli theory. Further, in Table-2, it is observed that present results are in good agreement with those reported by Reddy [7]. Small difference in results is observed for higher values of

Table-2 : Comparison of Non-dimensional Maximum Center Deflection $\left[\hat{w} = 10^2 \times w \left(\frac{EI}{qL^4}\right)\right]$ in Simply Supported - Simply Supported Uniform Beam

L/h	μ	Reddy [7]		Present Result		% of Difference	
		EBT	TBT	EBT	TBT	EBT	TBT
100	0.0	1.3130	1.3134	1.3021	1.3025	0.8317	0.8322
	0.5	1.3809	1.3813	1.3646	1.3650	1.1818	1.1822
	1.0	1.4487	1.4492	1.4271	1.4275	1.4924	1.4994
	1.5	1.5165	1.5170	1.4896	1.4900	1.7751	1.7818
	2.0	1.5844	1.5849	1.5521	1.5525	2.0399	2.0462
	2.5	1.6522	1.6528	1.6146	1.6150	2.2770	2.2888
	3.0	1.7201	1.7207	1.6771	1.6775	2.5010	2.5123
	3.5	1.7879	1.7886	1.7396	1.7400	2.7026	2.7189
	4.0	1.8558	1.8565	1.8021	1.8025	2.8947	2.9103
	4.5	1.9236	1.9244	1.8646	1.8650	3.0682	3.0882
5.0	1.9914	1.9923	1.9271	1.9275	3.2299	3.2540	
10	0.0	1.3130	1.3483	1.3021	1.3343	0.8317	1.0398
	0.5	1.3809	1.4210	1.3646	1.3968	1.1818	1.7044
	1.0	1.4487	1.4937	1.4271	1.4593	1.4924	2.3043
	1.5	1.5165	1.5664	1.4896	1.5218	1.7751	2.8486
	2.0	1.5844	1.6391	1.5521	1.5843	2.0399	3.3445
	2.5	1.6522	1.7118	1.6146	1.6468	2.2770	3.7983
	3.0	1.7201	1.7845	1.6771	1.7093	2.5010	4.2152
	3.5	1.7879	1.8572	1.7396	1.7718	2.7026	4.5994
	4.0	1.8558	1.9299	1.8021	1.8343	2.8947	4.9547
	4.5	1.9236	2.0026	1.8646	1.8968	3.0682	5.2841
5.0	1.9914	2.0754	1.9271	1.9593	3.2299	5.5951	

nonlocal parameter and thick beams. This is attributed to the shear force effect in thick beams and Rayleigh-Ritz method approximation. In CF beam deflection is decreasing with increasing non-local parameter. Thus in case of CF beam stiffness is directly proportional to the non-local parameter. However, in case of SS beam maximum deflec-

tion is increasing with increase in non-local parameter. Thus in case of SS beam the beam stiffness is inversely proportional to the non-local parameter. This is due to small scale effect at molecular level.

Table-3 : Non-dimensional Maximum Deflection $\left[\hat{w} = 10^2 \times w \left(\frac{EI}{qL^4} \right) \right]$ in Clamped - Simply Supported Uniform Beam				
μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	0.5389	0.5421	0.5849	0.7094
0.5	0.5565	0.5596	0.6030	0.7288
1.0	0.5742	0.5771	0.6211	0.7482
1.5	0.5918	0.5946	0.6392	0.7676
2.0	0.6094	0.6121	0.6573	0.7870
2.5	0.6271	0.6296	0.6754	0.8063
3.0	0.6447	0.6471	0.6935	0.8257
3.5	0.6623	0.6646	0.7116	0.8451
4.0	0.6800	0.6821	0.7297	0.8645
4.5	0.6976	0.6996	0.7478	0.8839
5.0	0.7152	0.7171	0.7659	0.9033

Non-dimensional maximum deflections are computed for the uniform beam with CS boundary condition. Results are listed in Table-3. In case of CS uniform beam deflection is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory. This is due to the fact that beam stiffness for CS boundary condition is inversely proportional to the nonlocal parameter. This is due to small scale effect at molecular level. For the increase of nonlocal parameter from 0 to 5 there is an increase of 51 percent, increase of 34 percent and decrease of 20 percent in maximum deflections for SS, CS and CF uniform beams, respectively. Nonlocal effect is found to be in increasing order for CF, CS and SS boundary conditions.

Vibration of Beam

Nonlocal fundamental frequencies for SS, CS and CF beams are computed as mentioned in Eqns. (39,47). Beam configurations are assumed as mentioned in numerical example of sub section Bending of beam. The fundamental frequencies for SS beam are listed Table-4. From this table one could observe that present results are in good agreement with those reported in Reddy [7]. Small difference

Table-4 : Comparison of Non-dimensional Fundamental Natural Frequencies $\left[\bar{\omega} = \omega_1 L^2 \sqrt{\frac{\rho A}{EI}} \right]$ in Simply Supported - Simply Supported Uniform Beam							
L/h	μ	Reddy [7]		Present Result		% of Difference	
		EBT	TBT	EBT	TBT	EBT	TBT
100	0.0	9.8696	9.8683	9.8745	9.8706	0.05005	0.02331
	0.5	9.6347	9.6335	9.4297	9.4266	2.12752	2.14792
	1.0	9.4159	9.4147	9.0543	9.0518	3.84010	3.85472
	1.5	9.2113	9.2101	8.7306	8.7286	5.21848	5.22850
10	0.0	9.8696	9.7454	9.8745	9.7482	0.05005	0.02842
	0.5	9.6347	9.5135	9.4297	9.3279	2.12752	1.95081
	1.0	9.4159	9.2973	9.0543	8.9708	3.84010	3.51188
	1.5	9.2113	9.0953	8.7306	8.6612	5.21848	4.77279

in results is observed for higher values of nonlocal parameter and thick beams. This is attributed to the shear force effect in thick beams. In case of SS beam natural frequency is observed to be decreasing with increase in non-local parameter. Thus in case of SS beam the beam stiffness is inversely proportional non-local parameter.

Non-dimensional fundamental frequencies are computed for the uniform beam with CS and CF boundary conditions. Results are listed in Tables-5 to 6, respectively. In case of CS uniform beam fundamental frequency is observed to be decreasing with increase in non-local parameter and inclusion of Timoshenko beam theory. This is due to the fact that beam stiffness for CS boundary condition is inversely proportional to the nonlocal parameter. While, in case of CF uniform beam fundamental frequency is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory. This is due to the fact that beam stiffness for CF boundary condition is directly proportional to the nonlocal parameter.

For the increase of nonlocal parameter from 0 to 5 there is decrease of 26 percent and increase of 5 percent in natural frequencies for CS and CF uniform beams, respec-

tively. Nonlocal effect is found to be in increasing order for CF and CS boundary conditions.

Buckling of Column

Nonlocal critical buckling loads for SS, CS and CF columns are computed as mentioned in Eqns. (54, 63). Column configurations are assumed as mentioned in numerical example of sub section Bending of beam. The SS column results are listed Table-7. From this table one could observe that present results are in good agreement with those reported in Reddy [7]. Small difference in results is observed for higher values of nonlocal parameter and thick beams. This is attributed to the shear force effect in thick beams. In case of SS column critical buckling load is observed to be decreasing with increase in non-local parameter. Thus in case of SS beam the beam stiffness is found to be inversely proportional to non-local parameter.

Non-dimensional critical buckling loads are computed for the uniform beam with CS and CF boundary conditions. Results are listed in Tables-8 to 9, respectively. In case of CS uniform beam critical buckling load is observed to be decreasing with increase in non-local parameter and inclusion of Timoshenko beam theory. This is due to the fact that beam stiffness for CS boundary condition is

Table-5 : Non-dimensional Fundamental Frequencies $\left[\bar{\omega} = \omega_1 L^2 \sqrt{\frac{\rho A}{EI}} \right]$ in Clamped - Simply Supported Uniform Beam

μ	EBT L/h = 100	TBT L/h = 100	TBT L/h = 10	TBT L/h = 5
0.0	15.4252	15.4141	14.9345	13.7532
0.5	14.6405	14.6330	14.2450	13.2736
1.0	14.0137	14.0102	13.6871	12.8719
1.5	13.4968	13.4966	13.2223	12.5287
2.0	13.0674	13.0627	12.8268	12.2313
2.5	12.6798	12.6896	12.4848	11.9707
3.0	12.3650	12.3646	12.1856	11.7405
3.5	12.0804	12.0784	11.9213	11.3529
4.0	11.7997	11.5975	11.6861	11.1891
4.5	11.5673	11.3940	11.4758	11.0421
5.0	11.3579	11.2108	11.2869	10.9103

Table-6: Non-dimensional Fundamental Frequencies $\left[\bar{\omega} = \omega_1 L^2 \sqrt{\frac{\rho A}{EI}} \right]$ in Clamped - Free Uniform Beam

μ	EBT L/h = 100	TBT L/h = 100	TBT L/h = 10	TBT L/h = 5
0.0	3.5168	3.5160	3.4981	3.4428
0.5	3.5322	3.5312	3.5137	3.4593
1.0	3.5479	3.5468	3.5297	3.4761
1.5	3.5639	3.5627	3.5461	3.4933
2.0	3.5803	3.5790	3.5629	3.5110
2.5	3.5972	3.5957	3.5800	3.5291
3.0	3.6144	3.6128	3.5976	3.5477
3.5	3.6321	3.6303	3.6157	3.5667
4.0	3.6503	3.6483	3.6342	3.5863
4.5	3.6689	3.6667	3.6532	3.6064
5.0	3.6880	3.6856	3.6727	3.6272

Table-7 : Comparison of Non-dimensional Critical Buckling Loads $\left[P = P_{cr} \times \left(\frac{L^2}{EI} \right) \right]$ in Simply Supported - Simply Supported Uniform Beam							
L/h	μ	Reddy [7]		Present Result		% of Difference	
		EBT	TBT	EBT	TBT	EBT	TBT
100	0.0	9.8696	9.8671	9.8696	9.8621	0.0001	0.0511
	0.5	9.4055	9.4031	9.0094	9.0033	4.2110	4.2523
10	0.0	9.8696	9.6227	9.8696	9.6227	0.0001	0.0002
	0.5	9.4055	9.1701	9.0094	8.7789	4.2110	4.2658

Table-8 : Non-dimensional Critical Buckling Loads $\left[P = P_{cr} \times \left(\frac{L^2}{EI} \right) \right]$ in Clamped - Simply Supported Uniform Column				
μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	21.7050	21.5844	20.0970	16.7875
0.5	17.9782	17.8799	16.7247	14.0205
1.0	15.4984	15.4102	14.4570	12.1371
1.5	13.7005	13.6168	12.8026	10.7529
2.0	12.3238	12.2418	11.5307	9.6893
2.5	11.2289	11.1467	10.5161	8.8446
3.0	10.3332	10.2498	9.6845	8.1552
3.5	9.5844	9.4990	8.9881	7.5794
4.0	8.9475	8.8596	8.3949	7.0891
4.5	8.3980	8.3073	7.8827	6.6648
5.0	7.9183	7.8246	7.4353	6.2931

Table-9 : Non-dimensional Critical Buckling Loads $\left[P = P_{cr} \times \left(\frac{L^2}{EI} \right) \right]$ in Clamped - Free Uniform Column				
μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	2.4897	2.4672	2.4488	2.3940
0.5	2.5481	2.5301	2.5104	2.4517
1.0	2.6105	2.5971	2.5759	2.5131
1.5	2.6774	2.6688	2.6460	2.5785
2.0	2.7493	2.7457	2.7211	2.6484
2.5	2.8269	2.8286	2.8019	2.7233
3.0	2.9111	2.9183	2.8892	2.8039
3.5	3.0027	3.0157	2.9840	2.8911
4.0	3.1029	3.1221	3.0873	2.9856
4.5	3.2132	3.2391	3.2006	3.0887
5.0	3.3354	3.3685	3.3258	3.2019

inversely proportional to the nonlocal parameter. While, in case of CF uniform beam critical buckling load is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory. This is due to the fact that beam stiffness for CF boundary condition is directly proportional to the nonlocal parameter.

For the increase of nonlocal parameter from 0 to 5 there is an decrease of 63 percent and increase of 5 percent in critical buckling loads for CS and CF uniform columns, respectively. Nonlocal effect is found to be in increasing order for CF and CS boundary conditions.

Bending of Tapered Beam

Present flexural response computation is extended to tapered beams Non-dimensional maximum deflections for SS, CS and CF boundary conditions are being computed and listed in Tables-10 to 12, respectively. $I(x)$, $A(x)$ and $h(x)$ are integrated from 0 to L and the integrated values are used in the computation of tapered beam.

From Tables 10-12 following observations are made. In case of SS and CS tapered beam deflection is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory. In case of CF tapered beam deflection is observed to be decreasing with

Table-10 : Non-dimensional Maximum Center Deflection $\left[\hat{w} = 10^2 \times w \left(\frac{EI}{qL^4} \right) \right]$ **in Simply Supported - Simply Supported Tapered Beam**

μ	EBT L/h = 100	TBT L/h = 100	TBT L/h = 10	TBT L/h = 5
0.0	0.8703	0.8800	0.8926	0.9590
0.5	0.9126	0.9225	0.9345	1.0091
1.0	0.9546	0.9650	0.9764	1.0428
1.5	0.9968	1.0075	1.0183	1.0847
2.0	1.0384	1.0500	1.0602	1.1266
2.5	1.0803	1.0924	1.1021	1.1684
3.0	1.1222	1.1349	1.1440	1.2103
3.5	1.1642	1.1774	1.1859	1.2522
4.0	1.2061	1.2199	1.2278	1.2941
4.5	1.2480	1.2623	1.2697	1.3360
5.0	1.2899	1.3048	1.3117	1.3778

Table-12 : Non-dimensional Maximum Deflection $\left[\hat{w} = w \left(\frac{EI}{qL^4} \right) \right]$ **in Clamped - Free Tapered Beam**

μ	EBT L/h = 100	TBT L/h = 100	TBT L/h = 10	TBT L/h = 5
0.0	0.1059	0.1059	0.1069	0.1098
0.5	0.1040	0.1040	0.1050	0.1078
1.0	0.1021	0.1021	0.1030	0.1059
1.5	0.1001	0.1001	0.1011	0.1040
2.0	0.0982	0.0982	0.0992	0.1020
2.5	0.0963	0.0963	0.0972	0.1001
3.0	0.0943	0.0943	0.0953	0.0982
3.5	0.0924	0.0924	0.0934	0.0962
4.0	0.0905	0.0905	0.0914	0.0943
4.5	0.0885	0.0885	0.0895	0.0924
5.0	0.0866	0.0866	0.0876	0.0905

Table-11 : Non-dimensional Maximum Deflection $\left[\hat{w} = 10^2 \times w \left(\frac{EI}{qL^4} \right) \right]$ **in Clamped - Simply Supported Tapered Beam**

μ	EBT L/h = 100	TBT L/h = 100	TBT L/h = 10	TBT L/h = 5
0.0	0.3308	0.3858	0.4044	0.4727
0.5	0.3460	0.3975	0.4167	0.4857
1.0	0.3611	0.4092	0.4290	0.4988
1.5	0.3762	0.4209	0.4414	0.5118
2.0	0.3913	0.4326	0.4537	0.5248
2.5	0.4064	0.4443	0.4661	0.5378
3.0	0.4215	0.4561	0.4784	0.5508
3.5	0.4366	0.4678	0.4907	0.5638
4.0	0.4518	0.4795	0.5031	0.5768
4.5	0.4669	0.4912	0.5154	0.5999
5.0	0.4820	0.5029	0.5278	0.6029

increase in non-local parameter and inclusion of Timoshenko beam theory. For L/h=100 EBT and TBT yield same results because of cumulative effect of tapering, CF boundary condition and nonlocal elasticity.

For the increase of nonlocal parameter from 0 to 5 there is an increase of 47 percent, increase of 45 percent and decrease of 19 percent in maximum deflections for SS, CS and CF tapered beams, respectively. Nonlocal effect is found to be in increasing order for CF, CS and SS boundary conditions.

Vibration of Tapered Beam

Present beam vibration computation is extended to tapered beams. Non-dimensional fundamental frequencies for SS, CS and CF boundary conditions are being computed and listed in Tables-13 to 15, respectively. $I(x)$, $A(x)$ and $h(x)$ are integrated from 0 to L and the integrated values are used in the computation of tapered beam.

From these Tables 13-15 following observations are made. In case of SS and CS tapered beams vibration frequency is observed to be decreasing with increase in non-local parameter and inclusion of Timoshenko beam theory. In case of CF tapered beam vibration frequency is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory. For the increase of nonlocal parameter from 0 to 5 there is decrease of 26 percent, decrease of 27 percent and increase of 3 percent in maximum deflections for SS, CS and CF tapered beams, respectively.

Table-13 : Non-dimensional Fundamental Frequencies $\left[\bar{\omega} = \omega_1 L^2 \sqrt{\frac{\rho A}{EI}}\right]$ in Simply Supported - Simply Supported Tapered Beam

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	9.8691	9.8634	9.3622	8.1356
0.5	9.4244	9.4201	8.9847	7.8933
1.0	9.0491	9.0459	8.6606	7.6770
1.5	8.7255	8.7231	8.3775	7.4821
2.0	8.4419	8.4401	8.1267	7.3050
2.5	8.1900	8.1887	7.9021	7.1428
3.0	7.9640	7.9630	7.6990	6.9934
3.5	7.7593	7.7587	7.5139	6.8551
4.0	7.5726	7.5722	7.3442	6.7264
4.5	7.4012	7.4010	7.1877	6.6063
5.0	7.2431	7.2430	7.0463	6.4936

Table-15 : Non-dimensional Fundamental Frequencies $\left[\bar{\omega} = \omega_1 L^2 \sqrt{\frac{\rho A}{EI}}\right]$ in Clamped - Free Tapered Beam

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	2.8532	2.8390	2.7163	2.4163
0.5	2.8618	2.8476	2.7239	2.4220
1.0	2.8706	2.8562	2.7317	2.4277
1.5	2.8796	2.8650	2.7395	2.4336
2.0	2.8887	2.8739	2.7475	2.4395
2.5	2.8979	2.8830	2.7554	2.4455
3.0	2.9073	2.8922	2.7637	2.4515
3.5	2.9168	2.9016	2.7721	2.4577
4.0	2.9265	2.9111	2.7805	2.4639
4.5	2.9364	2.9208	2.7891	2.4702
5.0	2.9464	2.9307	2.7978	2.4766

Table-14 : Non-dimensional Fundamental Frequencies $\left[\bar{\omega} = \omega_1 L^2 \sqrt{\frac{\rho A}{EI}}\right]$ in Clamped - Simply Supported Tapered Beam

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	14.6927	14.6914	13.6500	11.2741
0.5	13.9265	13.9236	13.0368	10.9454
1.0	13.3183	13.3077	12.5312	10.6574
1.5	12.8128	12.7967	12.1033	10.4019
2.0	12.3825	12.3624	11.7340	10.1727
2.5	12.0097	11.9866	11.4105	9.9656
3.0	11.6822	11.6569	11.1237	9.7771
3.5	11.3912	11.3644	10.8671	9.6044
4.0	11.1304	11.1025	10.6356	9.4455
4.5	10.8949	10.8662	10.4253	9.2986
5.0	10.6809	10.6517	10.2333	9.1622

Buckling of Tapered Column

Non-dimensional critical buckling loads for SS, CS and CF tapered columns are being computed and listed in Tables-16 to 18, respectively. $I(x)$, $A(x)$ and $h(x)$ are integrated from 0 to L and the integrated values are used in the computation of tapered column.

From Tables 16-18 following observations are made. In case of SS and CS tapered columns critical buckling load is observed to be decreasing with increase in non-local parameter and inclusion of Timoshenko beam theory. In case of CF tapered column critical buckling load is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory.

For the increase of nonlocal parameter from 0 to 5 there is decrease of 49 percent, decrease of 63 percent and increase of 38 percent in maximum deflections for SS, CS and CF tapered columns, respectively. Nonlocal effect is found to be in increasing order for CF, CS and SS boundary conditions. From the present computation it is found that nonlocal elasticity has significant contribution for lower L/h ratios. ($L/h < 100$). Thus Timoshenko beam theory should be included in the analysis.

Nonlocal effect is found to be in increasing order for CF, CS and SS boundary conditions.

Table-16 : Non-dimensional Critical Buckling
Loads $\left[P = P_{cr} \times \left(\frac{L^2}{EI} \right) \right]$ in
Simply Supported - Simply Supported Tapered Column

μ	EBT L/h = 100	TBT L/h = 100	TBT L/h = 10	TBT L/h = 5
0.0	15.8390	14.4212	13.7967	13.3877
0.5	14.3026	13.1551	13.4934	12.2232
1.0	13.0833	12.1320	12.4406	11.2782
1.5	12.0856	11.2830	11.5675	10.4917
2.0	11.2502	10.5642	10.8284	9.8243
2.5	10.5379	9.9457	10.1925	9.2490
3.0	9.9218	9.4064	9.6383	8.7467
3.5	9.3823	8.9311	9.1499	8.3036
4.0	8.9053	8.5083	8.7155	7.9092
4.5	8.4798	8.1292	8.3261	7.5553
5.0	8.0974	7.7869	7.9746	7.2356

Table-18 : Non-dimensional Critical Buckling
Loads $\left[P = P_{cr} \times \left(\frac{L^2}{EI} \right) \right]$ in
Clamped - Free Tapered Column

μ	EBT L/h = 100	TBT L/h = 100	TBT L/h = 10	TBT L/h = 5
0.0	3.1462	3.1234	3.1027	3.0578
0.5	3.2291	3.2047	3.1837	3.1376
1.0	3.3176	3.2915	3.2701	3.2228
1.5	3.4124	3.3845	3.3627	3.3139
2.0	3.5144	3.4843	3.4621	3.4118
2.5	3.6244	3.5920	3.5693	3.5173
3.0	3.7436	3.7086	3.6853	3.6315
3.5	3.8734	3.8354	3.8116	3.7557
4.0	4.0155	3.9741	3.9496	3.8914
4.5	4.1721	4.1268	4.1014	4.0406
5.0	4.3460	4.2960	4.2697	4.2059

Table-17 : Non-dimensional Critical Buckling
Loads $\left[P = P_{cr} \times \left(\frac{L^2}{EI} \right) \right]$ in
Clamped - Simply Supported Tapered Column

μ	EBT L/h = 100	TBT L/h = 100	TBT L/h = 10	TBT L/h = 5
0.0	29.8530	29.7510	27.9463	23.8585
0.5	14.7636	24.6441	23.2081	19.8639
1.0	21.3875	21.2514	20.0401	17.1508
1.5	18.9459	18.7940	17.7383	15.1903
2.0	17.0811	16.9136	15.9747	13.7113
2.5	15.6017	15.4189	14.5722	12.5507
3.0	14.3950	14.1966	13.4256	11.6063
3.5	13.3891	13.1752	12.4679	10.8146
4.0	12.5364	12.3067	11.6541	10.1364
4.5	11.8035	11.5576	10.9527	9.5463
5.0	11.1664	10.9039	10.3410	9.0270

Conclusions

Effect of nonlocal parameter on the structural response is sensitive to the applied boundary conditions and Timoshenko beam theory.

In case of SS and CS beam deflection is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory. In case of CF uniform beam deflection is observed to be decreasing with increase in non-local parameter and inclusion of Timoshenko beam theory.

In case of SS and CS beams vibration frequencies are observed to be decreasing with increase in non-local parameter and inclusion of Timoshenko beam theory. In case of CF beam vibration frequency is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory.

In case of SS and CS columns critical buckling loads are observed to be decreasing with increase in non-local parameter and inclusion of Timoshenko beam theory. In case of CF column critical buckling load is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory.

Effect of nonlocal parameter is larger on bending and buckling than in vibration of beams. Effect of nonlocal parameter in case of CF boundary condition is substantially less than those for SS and CS boundary conditions. Further, effect of nonlocal parameter in case of CF boundary condition is opposite in nature as compared to those for SS and CS boundary conditions.

Nonlocal elasticity has significant contribution for lower L/h ratios and Timoshenko beam theory should be included in the analysis.

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