TECHNICAL NOTE

A COMPARATIVE STUDY OF PROBABILISTIC STRUCTURAL SAFETY ANALYSIS OF A TITANIUM PRESSURE VESSEL

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Abstract

Structural Reliability (Safety Index) is a very important factor of any aerospace product. Until recently, a design was considered robust if all the variables that affected its life had been accounted for and brought under control. The meaning of robustness is shifting. Designer and engineers have traditionally handled variability with safety factors. Some safety factors are derived from observation and analysis, and many cases it used to be pure guesswork. In those cases, the bigger the guess, the bigger the risk, the bigger the safety factor, resulting over designed product. Safety factor cannot, by themselves, guarantee satisfactory performance and they do not provide sufficient information to achieve optimal use of available resources. In this paper pressure vessel made of titanium alloy is considered for structural reliability study. Structural Safety Index is evaluated using uncertainty. Various statistical methods like mean value and moment methods are discussed. Simulation techniques for evaluation of probability of failure are also discussed. Response surface methodology which helps in solving many complex structural problems has been used. Finally, comparative studies have been made for various techniques. These techniques will be useful for Reliability design evaluation.

***Keywords**: Safety Index, Uncertainty, Structural Reliability, Probability of Failure, Advanced First Order Second Moment Method (AFOSM)*

Introduction

Probabilistic structural design evaluation is fast growing in aerospace engineering. When all uncertainties like variability in material properties, geometry and loads are considered in design, the product will have better reliability compared to deterministic design. In this paper a Titanium spherical nitrogen gas bottle is considered for reliability study. This nitrogen gas bottle is used in a space vehicle. The bottle is charged with high pressure nitrogen gas. This high-pressure gas is used for pneumatic actuation to drive turbine for power generation and also used for pressurization of fuel chambers. The safety and reliability are the prime requirements for these aerospace products. These bottles designed as per pressure vessel code are excessively used in various aerospace vehicles successfully. Data generated during manufacturing and operation have been analyzed systematically which is discussed in this paper. Structural reliability evaluated using various techniques are compared. Response surface methodology is used to establish multiple regression relation for hoop stress. Hoop stress data generated using ANSYS for a simple full factorial experiment and structural reliability (safety index) is evaluated using regression equation. Simulation techniques used for the evaluation of the failure probability are discussed in this paper.

Nomenclature

g (x)	= performance function
HS	= hoop stress
p	= pressure
p_f	= probability of failure
R_i	= internal radius
S	= ultimate strength of material
t	= thickness
u & U	= a vector of statistically independent random variables with zero mean and unit standard deviation
V_{pf}	= variance

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β = safety index
$\beta_0, \beta_1, \beta_2$ = regression coefficient
μ = mean
σ = standard deviation
$\varnothing$ = normal Pdf
Φ = normal Cdf

Limit State function: a set defined by locus of points, g(x) = 0,

Safety Index: β, the Safety Index, is defined as the scalar distance in standard normal space $\beta = \dfrac{\mu_G}{\sigma_G}$

Probabilistic Structural Safety

Structural analysis and design have traditionally been based in deterministic methods. However, uncertainties in the loads, strength and in the modeling of the systems require that methods based on probabilistic techniques in a number of situations have to be used. A structure is usually required to have a satisfactory performance in the expected life time, i.e. it is required that it does not collapse or become unsafe and that it fulfills certain functional requirements.

The probabilistic safety of structural system can be defined as the probability the structure under consideration has a proper performance throughout its life time. Reliability methods are used to estimate the probability of failure.

The reliability estimated as a measure of the safety of a structure can be used in design process. A lower level of the reliability can be used as a constraint in an optimal design problem.

Generally the steps involve in structural safety analysis are:

- Select a target safety (reliability) level.
- Identify the significant failure modes of the structure.
- Formulate failure functions (limit state functions) corresponding to each component in the failure modes.
- Identify the stochastic variables and the deterministic parameters in the failure functions. Further specify the distribution types and statistical parameters for the stochastic variables and the dependencies between them.
- Estimate the safety (reliability).
- Compare with the target reliability.
- Evaluate safety result by performing sensitivity analyses.

Typical failure modes to be considered in a safety analysis of a structural system are yielding, buckling, fatigue and deformations.

The failure modes (Limit States) are generally divided in following limit states.

Ultimate Limit States

Ultimate limit states correspond to the maximum load carrying capacity which can be related to formation of a mechanism in the structure, excessive plasticity, rupture due to fatigue and instability.

Conditional Limit States

Conditional limit states correspond to the load-carrying capacity if a local part of the structure has failed. A local failure can be caused by an accidental action or by fire. The conditional limit states can be related to e.g. formation of a mechanism in the structure, exceedance of the material strength or instability (buckling).

Serviceability Limit States

Serviceability limit states are related to normal use of the structure, e.g. excessive deflection, local damage and excessive vibrations.

The uncertainty modeled by stochastic variables can be divided in the following groups:

Physical Uncertainty: or inherent uncertainty is related to the randomness of a quantity.

Measurement Uncertainty: is the uncertainty caused by imperfect measurements.

Statistical Uncertainty: is due to limited sample size of observed quantities.

Model Uncertainty: is the uncertainty related to imperfect knowledge or idealizations of the mathematical model used.

Generally, methods to measure the safety (reliability) of a structure can be divided in four groups.

- *Level I methods*: The uncertain parameters are modeled by one characteristic value, as for example in codes based on the partial safety factor concept.
- *Level II methods*: The uncertain parameters are modeled by mean values and standard deviations, and by the correlation coefficients between the stochastic variables. The (reliability) safety index method is an example of a level II method.
- *Level III methods*: The uncertain quantities are modeled by their joint distribution functions. The probability of failure is estimated as measure of the safety (reliability).
- *Level IV methods*: In these methods the consequences (cost) of failure are also taken into account and the risk is used as a measure the reliability.

Several techniques can be used to estimate the safety for level II&III methods, e.g.

- *FORM techniques*: In First Order Reliability Methods the limit state function is linearized and the reliability is estimated using level II or III methods.
- *SORM techniques*: In Second Order Reliability Methods a quadratic approximation to the failure function is determined and the probability of failure for the quadratic failure surface is determined and the probability of failure surface is estimated.
- *Simulation technique*: Samples of the stochastic variables are generated and the relative number of samples corresponding to failure is used to estimate the probability of failure.

Some of the above techniques have been used in this paper are discussed in subsequent Para.

FORM Technique

Moment Method

Mean Value Method [1], [2]

This method is commonly referred to as the mean value first order-second moment (MVFOSM, MVM or simply MV) method since it involves a first order expansion about the mean to estimate the first and second moments. MV method involves developing the Taylor series expansion of g(x) about the nominal or mean values of the individual random variables. The moments of the resulting approximating function are found based on these moments approximate statements can be made regarding the probability of failure.

$$g(x) = g(x_1, x_2, \dots x_n) \tag{1}$$

$$\approx g(x)\big|_{x=\bar{\mu}} + \sum_{i=1}^{n} \frac{\partial g(x)}{\partial x_i}\bigg|_{x=\bar{\mu}} (x_i - \mu_i)$$
$$+ \frac{1}{2!} \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 g(x)}{\partial x_i \partial x_j}\bigg|_{x=\bar{\mu}} (x_i - \mu_i)(x_j - \mu_j) + H.O.T \tag{2}$$

$$E[g(x)] \approx g(x)\big|_{x=\bar{\mu}} + \frac{1}{2!} \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 g(x)}{\partial x_i \partial x_j}\bigg|_{x=\bar{\mu}}$$

$$E\left[(x_i - \mu_i)(x_j - \mu_j)\right] \; or \; E[g(x)] = g(x)\big|_{x=\bar{\mu}} \tag{3}$$

Similarly

$$v[g(x)] = \sum_{i=1}^{n} v[x_i] \left(\frac{\partial g(g)}{\partial x_i}\bigg|_{x=\bar{\mu}}\right)^2 \tag{4}$$

Safety Index

$$\beta = \frac{\mu_G}{\sigma_G} \tag{5}$$

Advanced First-Order Second-Moment

Hasofer and Lind [3], [4], [8] proposed a method for evaluating the safety index (β), according to HL approach the constraint is linearized by using Taylor series expansion retaining up to the first order terms. The linearization point selected is that of maximum likelihood of occurrence and is known as the most probable failure point. This method is called Advanced First-Order Second-Moment (AFOSM) method. The most probable failure point is determined by transforming original random variables to normalized and independent set of reduced variables as shown in Fig.1.

$$U = \frac{x - x_\mu}{s_\sigma} \tag{6}$$

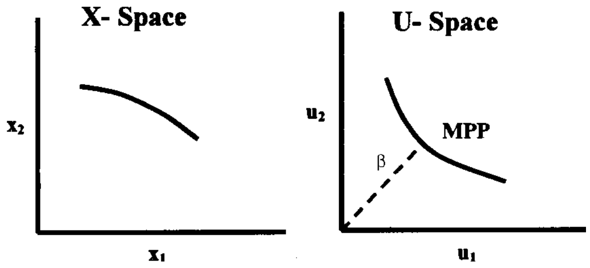

Fig.1 Transformation of Coordinate into Standard Space

The failure surface is mapped onto the corresponding failure surface in the reduced space. The point on this surface with minimum distance from the origin is the most probable failure point and the geometric distance to the origin is equal to the safety index β.

The failure surface is generally a nonlinear function, thus the point with minimum distance to the origin can be evaluated by solving the following optimization problem i.e

Minimize $(U^T U)^{1/2} = \beta$

Subject to g (U) = 0

Where g(U) is the failure function or limit state equation in reduced space. Using AFOSM, this optimization can be solved by using nonlinear optimization or iterative algorithms.

Simulation Technique

Simulation based algorithms [10], [11] were the first developed methods for reliability (probability of failure) analysis and included techniques such as Monte Carlo Simulation and Latin Hyper Cube sampling. Simulation techniques perform discrete series of numerical experiments to estimate the multi dimensional integral. These techniques are preferable for complex problems where accuracy is a primary concern since simulation tends to converge to theoretical solution. However, the major disadvantage is the large number of simulations required to obtain an accurate result. In this paper, Monte Carlo simulation, Latin Hyper Cube Sampling and Importance Sampling are discussed to provide comparative results. The results from Monte Carlo Simulation are treated as an exact solution.

Monte Carlo Simulation

Monte Carlo Simulation (MCS) is a traditional technique used in structural reliability analysis. It can be used to solve complex problems of implicit or explicit form for which accurate solutions are either impossible or extremely difficult to obtain. For example, a limit state involving multiple random variables that is evaluated with a nonlinear finite element model can be easily setup for solution with the Monte Carlo Simulation technique. The implementation of MCS in reliability analysis is relatively straight forward. However, the drawback is that it becomes computationally intensive when the sought probability of failure (p_f) or reliability Safety Index (β) is extremely low or high respectively. The relation between probability of failure and expected failures per simulation runs are related as given in Table-1.

The sampling cannot be expected to be ideal, and it is reasonable to expect 10 - 100 times the number of samples listed below for an accurate solution. In structural engineering, for a typical β of 3-4, this may lead to 100,000-10,000,000 samples.

The basic procedure of performing Monte Carlo Simulation is to define the Limit State Function and randomly generate values for the random variables using their probability distribution information. The probability of failure is calculated as below:

$$p_f = \frac{1}{N} \sum_{i=1}^{N} I_i \qquad (7)$$

Where

N = number of simulations

Table-1 : Relation of Failure Probability to Reliability Index are Required Samples

Probability of Failure (p_f)	Reliability Index (β)	Single Failure Ideally Occurs per Simulation
0.5	0	2
0.1	1.282	10
0.01	2.326	100
0.001	3.090	1000
0.0001	4.265	10000
0.00001	4.753	100000

I_i = indication function where $I_i = 1$ when failure occurs ($g < 0$) and $I_i = 0$ for survival ($g > 0$).

$$\beta = \phi^{-1} (1 - p_f) \quad (8)$$

$$R = 1 - \Phi(-\beta) \quad (9)$$

An estimate of the coefficient of variation of the MCS is given by

$$V_{pf} = \sqrt{\frac{(1 - p_f)}{N p_f}} \quad (10)$$

Importance Sampling

In 1986, Bucher proposed a focused sampling technique in which the sampling domain is focused on the Most Probable Point of Failure (MPP). This method is referred to as Importance Sampling (IS). This method results in a reduction of computational intensity. To identify the MPP, beta based reliability analysis algorithms are usually used.

$$p_f = \int I(g \leq 0) \frac{f_x(x)}{h_x(x)} . h_x(x) . dx \quad (11)$$

$$p_f = \frac{1}{N} \sum_{j=1}^{n} \left(I(g \leq 0) . \frac{f_x(x)}{h_x(x)} . h_x(x) \right) \quad (12)$$

Where

I = indicator, I equals 1 for failure and I equals 0 for survival.

$f_x(x)$ = probability density function evaluated at x, based on the original RV statistical parameters.

$h_x(x)$ = probability density function evaluated at x, based on MPP statistical parameters.

Importance sampling relies on the MPP and it will give poor result if the true MPP cannot be accurately located.

Latin Hyper Cube Sampling

The Latin Hyper Cube Sampling (LHS) method [10] is a technique for reducing a number of MCS simulation needed. It was first proposed by Mc Kay in 1979 and then further refined by Ronald L. Iman in 1981.

In Latin Hyper Cube, the samples are forced to be in interested region. This can be done by dividing standard normal probability density function into desired interval. The area under PDF curve for each interval is equal. Therefore, the divided Cumulative Density Function (CDF) can be developed.

The random number is generated by choosing standard normal CDF at each interval. The chosen CDF is then appropriately transformed onto basic domain according to desired distribution. Once all samples are generated, the set of random variables is organized by independently by uniformly selecting each of variables generated values such that the selected value must be used only once, this is depicted in Fig.2. The simulation steps are as follows :

- For each variable generate one point from each of the interval u_{ij}, j=1, 2,, m for variable i.
- The first point u_j^1 in the LHS sample is generated by sampling one value u_{ij}^1 from each axis u_i. The second point is generated in the same way, except that the values u_{ij}^1 are deleted from the sample. In this way 'm' points are generated.
- The probability of failure from this sample is estimated from

$$p_f = \frac{1}{m} \sum_{j=1}^{m} I\left[g(u^{\hat{j}}) \right] \quad (13)$$

- This procedure is repeated N times and the final estimate of p_f is

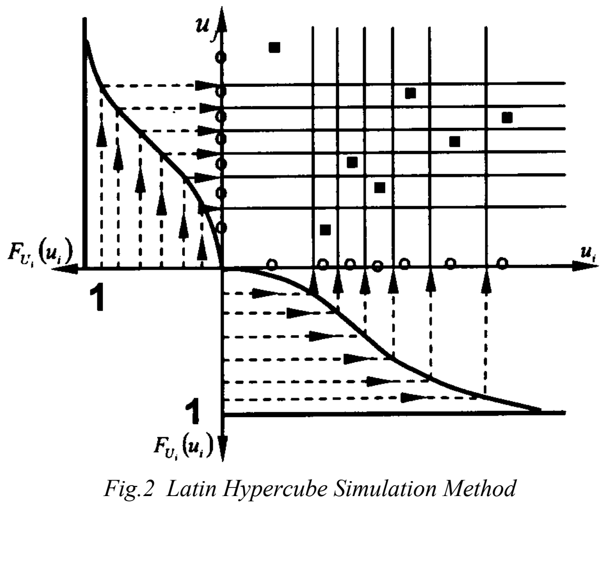

Fig.2 Latin Hypercube Simulation Method

$$p_f = \frac{1}{Nm} \sum_{k=1}^{N} \sum_{j=1}^{m} I\left[g(u^{kj}) \right] \quad (14)$$

Where u^{kj} is the realization number, $\hat{J}$ is the k[th] LHC sample.

There is no simple form for the standard error of this simulation but in general the standard error is of the magnitude $\frac{1}{mN}$ times the standard error of crude MCS.

Response Surface Method [5-9]

Time consuming simulations solely provide point wise information about input-output relations in the design space. To explore the entire design space and reduce computational burden, global approximation methods are applied. These approximations are also known as Meta model or Surrogate model. A Meta model replaces the true functional relationship with mathematical expression Y(x), the so-called response surface, which is much easier to compute. There are different Meta model types: Polynomial, Kriging, Radial basis function, neural network etc. They differ in the mathematical expressions that describe the output variables which Meta model type should be applied depends on the true input-output relationship. We use here the polynomial Meta model in any order, which is easy to employ and able to represent robust most global input-output-relationship for the response surface methodology [5]-[9]. For example, the Second Order polynomial can be explained.

$$Y = \beta_0 y_0 + \sum \beta_{1i} x_i + \sum \beta_{2i} x_i^2 + \sum \beta_{ij} x_i x_j \quad (15)$$

For the generation of the Meta model, that means computing the unknown factor β of the polynomial, an appropriate number of sampling points is needed. These so-called support points can be selected via different Design of Experiment Technique (DoE) in order to gain maximum information on the characteristics of the underlying relationship. For each point, the time consuming simulation of the model is performed to get the true value of the response variable Y. For the second order polynomial, $\frac{n^2}{2} + \frac{3n}{2} + 1$ model calculations are required, where n is the number of input variables. To fit the Meta model with the unknown factor β to the support points, the least square method or so called linear regression analysis is used.

The probability distribution of the output variable Y can be gained using the Monte Carlo Sampling with a virtual sample set based on the approximated Meta model. The computing time here is negligible compared to a time-consuming model calculation. The results are very accurate using a high virtual sampling size. But they are still stochastic and instable, because they depend up on the specific virtual sample set. Therefore, the method is only conditionally applicable for a robust design optimization considering reliability.

Illustration

In this paper the applicability of various methods has been discussed. A spherical titanium pressure vessel, which has already been developed and successfully used in the aerospace vehicles, have been considered for illustration. These pressure vessels are charged with either air, nitrogen or inert gas. High pressure gas is used for pneumatic actuation, power generation and pressurisation of fuel chambers. These pressure vessels are very important hardware. Safety and Reliability are the prime requirements. The probability of failure and safety index is evaluated and compared at various operating pressure.

Data Collection

Design data are collected from the approved design report. The deterministic design parameters are given in Table-2.

A systematic data collection is carried out during manufacturing at production centers, starting from raw material to final product. Material properties are taken from test certificates provided by suppliers, which covers chemical compositions, heat treatment details, ultimate tensile strength, yield strength and percentage elongation. Similarly, thickness mapping for wall thickness and inter-

Table-2 : Design Parameter

Radius (R_i)	149.5 mm
Wall Thickness (t)	8.0 mm
Working Pressure	35 MPa
Design Pressure	40 MPa
Material	Titanium Alloy
Construction	Welded
Type of Welding	Electron Beam
Weight	12 Kg

nal radius are carried out before welding of each spherical shell. Parameter variability observed are given in Table -3.

Analysis

Safety Index is evaluated for the pressure vessel considering the data of Table-2 and Table-3. Safety Index evaluated based on Moment method, Response Surface method and Simulation method are placed in Table-4, Table-5 and Table-6.

Table-3 : Variability Observed

	Pressure (MPa) (P)	Internal Radius (R_i) mm	Thickness (t) mm	Material Strength (S) MPa
Mean μ	36	149.5	7.91	860
Standard deviation σ	1.98	0.50	0.25	8.6

Moment Method

Table-4 : Safety Index - β

	Operating Pressure (MPa)		
	36	70	80
MVM	22.34	8.53	4.47
AFOSM	15.541	5.77	2.90
HL	15.541	5.72	2.83

Response Surface Method

Response Surface equation generated from DOE is given in Eq (16).

$$HoopStress = 26.2471 - 12.034p - 3.8708R_i + 93.3623t \quad (16)$$

Above Meta model is used for evaluation of safety index / probability of failure

Simulation Method

The Structural Reliability / Safety Index of the pressure vessel is very high as $\beta > 15$, hence the probability of failure of this vessel is extremely low. Such low value of failure probability needs large number of simulation runs.

Table-5 : Safety Index - β

	Operating Pressure (MPa)		
	36	70	80
MVM	16.18	4.36	0.89
AFOSM	16.27	4.37	0.87
HL	16.22	4.38	0.89

Simulation runs are made for operating pressure 70 MPa and 80 MPa as given in Table-6.

Results and Discussion

Various techniques have been used and compared in this paper for Safety Index analysis. The results are given in Table-4, 5 and 6. From the results, it can be said that the pressure vessel is safe at an operating pressure 36MPa as β is high. All the techniques are equally applicable. The Hasofer Lind method is accurate compare to other method. Response Surface Method is simple subject to domain validity. This is evident from the test result that at higher pressure of 80MPa, the RSM does not indicate encouraging value of β. The Crude Monte Carlo is accurate and

Table-6 : Probability of Failure by Simulation

Simulation	Operating Pressure	
	70 MPa	80 MPa
Crude Monte Carlo (Simulations run)		
100	*	*
1000	*	1×10^{-3}
10000	*	6×10^{-4}
100000	*	1×10^{-3}
Importance Sampling		
100	5.1×10^{-13}	5.3×10^{-4}
1000	6.6×10^{-13}	5.2×10^{-4}
10000	6.4×10^{-13}	5.2×10^{-4}
Latin Hyper Cune Sampling (N x m)		
(20 x 50) = 1,000	*	*
(200 x 50) = 10,000	*	*
(200 x 100) = 20,000	2×10^{-4}	0.0343
(50 x 100) = 5,000	2×10^{-4}	0.0334
* No Detection		

closely matches with other methods. Since β >15 it means the vessel is safe, hence same vessel can be operated at higher pressure or the weight can be reduced. The safety Index (β) versus probability of failure (pf) is given in Table-7.

Table-7 : Safety Index Vs Probability of Failure

Safety Index (b)	Probability of Failure P (f)
1	0.1587
2	0.0227
3	0.0013
4	0.3162 E-04
5	0.2859 E-06
6	0.9716 E-09
7	0.1254 E-11
8	0.6056 E-15

Conclusion

In this paper we examined a variety of methods used to evaluate the Safety Index / probability of failure. A general comparison of the accuracy of the Safety Index estimates and the conservatism of the estimation are discussed. The Hasofer Lind method is the most efficient and simple method. Crude Monte Carlo is also accurate enough but intensive computationally. The Importance Sampling method is far less efficient computationally. However, a low value of probability can be detected with less number of simulation runs. Latin Hyper Cube Sampling method found equally efficient and useful.

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