AN IMPROVED DYNAMIC CONDENSATION APPROACH

Vivekanand S. Kanade[1], S.M. Kamat[2], B.R. Jayalekshmi[3],
K. Badari Narayana[4] and Gajbir Singh[5]

Abstract

It is difficult to determine natural frequencies and modal shapes of a Dynamic system having large number of degrees of freedom (DOF) and, in addition, expensive. It is always desirable to reduce the DOF of the system. Of various methods Guyan's condensation and Component modal analysis are quite popular for reducing the problem size. The present work deals with Guyan's approach and improvements in its basic form to suit for dynamic problems. The Guyan's Reduction method ignores the mass in calculating the transformation matrix, hence for dynamic problems its accuracy is usually low. This problem is overcome, by adopting an iterative technique in which the inertia terms are improved by a linearization process. Another important aspect in the implementation of a condensation procedure is the selection of the primary and slave DOF. In the present work, a two step approach is attempted wherein, first, a primary DOF set is selected on the basis of energy method or Ritz vectors and later, the inertial contribution of the transformation matrix is improved through an iterative procedure. The effectiveness of the proposed method is illustrated with two examples.

Key Words: Condensation, Frequency, FEM, Master and slave DOF, Dynamic analysis

Notations

[C]	= damping matrix
DOF	= degrees of freedom
[D]	= dynamic stiffness matrix
FRF	= frequency response functions
{F}	= force vector
[K]	= stiffness matrix
$[\overline{K}]$	= generalized stiffness matrix
[M]	= mass matrix
$[\overline{M}]$	= generalized mass matrix
N	= size of matrices
[T]	= transformation matrix
{u}	= displacement vector
$\{\ddot{u}\}$	= acceleration vector
ω	= natural frequency
$[\phi]$ or $[\Phi]$	= eigen vector matrix
Λ	= eigen value

Subscripts

c	= connected DOF
i	= interior DOF
m	= master DOF
s	= slave DOF

Introduction

Finite Element Method is a popular tool for solving structural problems. Particularly in dynamics, the finite element solutions can approach close to the real situation within a wide frequency range. However, to make sure that the results have the necessary accuracy, the finite element model with finer meshing is required, which inturn leads to large number of degrees of freedom (DOF) in the system. As a consequence, the stiffness, mass and damping matrices will be of large dimensions, requiring

1 Wsatkims, Victor House, Third Floor, Gandhi Nagar, Bangalore-560 009, India
2 Structures Group, ISRO Satellite Centre, Airport Road, Bangalore-560 017, India
3 Department of Civil Engineering, NITK Surathkal, Surathkal-575 025, India
4 Tata Consultancy Services Ltd, GR Tech Park, White Field Road, Bangalore-560 066, India; Email : bnkantheti@yahoo.co.in
5 CADES Digitech Pvt Ltd, Anchorage No. 100/1, Richmond Road, Bangalore-560 025, India
Manuscript received on 27 Oct 2006; Paper reviewed, revised and accepted on 28 May 2007

This work is done as a part of first author's ME dissertation under the guidance of Badari Narayana and Kamat, while he was at ISRO Satellite Centre, Bangalore.

huge computer storage and solution time. The analysis may, therefore be carried out more efficiently if the unwanted DOF can be eliminated by some reduction procedure before solving the Eigen problem. Numerous reduction techniques have been proposed over the last three decades. The most popular reduction methods are Guyan Condensation method and Component Mode Synthesis.

The Guyan's reduction method ignores the mass (inertia effect) in calculating the transformation matrix. The relation between the primary and secondary DOF is therefore found by establishing the static relation between them. Hence the process is only exact for static problems. In order to reduce the mass matrix of the system, the same static relation is assumed to remain valid for the dynamic problems. Hence, for dynamic problems the accuracy is usually low. This problem is overcome, by adopting an iterative technique in which the inertia terms are improved by a linearization process.

Another important aspect in the implementation of a condensation procedure is about the selection of the master and slave DOFs. The selected master degrees should describe the lowest modes accurately. The difficulty is in a given problem to determine which and how many master DOF should be included in the analysis. Improper selection of primary DOF or an insufficient number of them would give erroneous result.

Investigations by Ki-Ook Kim and Young-Jae Choi [1] and Arnold et. al [2] have showed that by comparing energy distribution over the DOF /or by using Ritz vectors the primary DOF can be selected effectively. As a principle, in system condensation the energy should be preserved in each eigen mode. Hence, the reduced stiffness and mass matrices conserve the strain energies. Nevertheless the equation of motion in the reduced subspace does not satisfy the equilibrium of the original system [3]. The incorrect energy distribution over the retained degrees of freedom is the major source of error in a condensation process.

Arnold et. al [2] showed that a sequential elimination of the secondary degrees of freedom is based on the highest ratio of diagonal terms of the structural matrices (K_{ii}/M_{ii}). It is shown [2] that the lowest one-third of the eigen values of the condensed system will be within the engineering accuracy of 5%. The minimisation of the energy unbalance has been successfully applied to the structural optimization of dynamic systems [4-6] using the primary set of DOF dominating the energy distribution. Because there are no priori exact eigenmodes available, Ritz vectors are calculated for the energy estimation. The energy distribution varies from mode to mode and so does the primary set. For proper selection of the DOF, two approaches are considered:

- A sequential selection, where in a union of finite number of DOF with the largest energy in each mode.
- A row sum selection, the components of energy distribution matrix along the row is added. The DOF with the largest sum are selected for the primary set.

With the mass normalisation of mode shapes, not only the energy of the DOF in each eigen mode (column sum) but also the energy of each DOF wherein the row sum becomes (for the whole eigen modes) unity. Hence, the partial row sum of a DOF indicates the importance of DOF to the modes of interest. So this DOF can be used as a primary DOF. Whereas, the partial column sum shows the contribution of the selected set in each mode and can be used to predict the solution accuracy [7-9].

In the present work, a two step approach is attempted by judiciously combining the approaches given in Refs. [1] and [10] to get accurate solution for a given eigen value problem viz; a) first, master DOF set is selected based on the energy method or using Ritz vectors and b) inertial contribution of these master DOF is improved through an iterative procedure. The effectiveness of the proposed method is illustrated with two examples a) An 'L' shaped beam and b) Dynamic response of a floating raft isolation system. The results of these examples are compared with the solutions given by Ref. [10] and [13].

Theoretical Basis

In the following sections, the mathematical expressions related to the condensation process are briefly described.

Energy Method [1] for Selection of Primary DOF

The method is based on the energies associated with the degrees of freedom in the eigenmodes of structural systems. For the energy estimation Ritz vectors (which are calculated using the stiffness and mass matrices) are used. The energies added through the modes of the energy distribution matrix can be used as an effective guideline for the retained degrees of freedom as the primary DOF in the analysis. Another approach is the sequential selec-

tion, in which a finite number of DOF with the largest energy are taken in each mode. The energy of the selected DOF or the column sum of the matrix can be used to predict the accuracy of solution in each mode.

Energy for Modes and Degrees of Freedom

In finite element analysis a general eigenproblem for undamped free vibration is given as

$$[K]\ \{\Phi\} = \lambda\,[M]\{\Phi\} \tag{1}$$

where [K] and [M] are stiffness and mass matrices. The Eigen pair (Eigen value and Eigen vector) is denoted as λ and $\{\Phi\}$. Assuming the Eigen problem given by Eqn. (1) has N eigenmodes, these can be written in compact form as

$$[K]\,[\Phi] = [M]\,[\Phi]\,[\Lambda] \tag{2}$$

where $[\Lambda]$ is a diagonal matrix for the eigen values. Pre-multiplying Eqn. (2) by $[\Phi]^T$ gives

$$[\Phi]^T\,[K]\,[\Phi] = [\Phi]^T\,[M]\,[\Phi]\,[\Lambda] \text{ or } [\bar{K}] = [\bar{M}]\,[\Lambda] \tag{3}$$

where $[\bar{K}]$ and $[\bar{M}]$ are generalised stiffness and mass matrices, respectively. Eqn. (3) shows not only the orthogonality of the Eigen modes but also the equilibrium of the strain and kinetic energies in each mode. Eqn. (3) is equivalent to Eqn (2). A set of L (< N) vectors satisfying the orthogonality may not be the eigenvectors of the structure (Eqn. (3) is only a necessary condition for the vectors). Introducing cross multiplication of matrices [1] as given below

$$[C] = [A] \otimes [B] \quad \text{or} \quad c_{ij} = a_{ij}\ b_{ij} \tag{4}$$

Equation (2) can be written as

$$[\Phi] \otimes [K]\,[\Phi] = [\Phi] \otimes [M]\,[\Phi]\,[\Lambda] \tag{5}$$

Equation (5) shows the equilibrium in terms of energies which states that each degree of freedom should be in equilibrium with respect to the strain and kinetic energies in each eigenmode. For illustration, generalised mass matrix will be considered first

$$[G] = [\Phi] \otimes [M]\,[\Phi] \text{ or } g_{ij} = \sum_{k=1}^{N} \Phi_{ij}\, M_{ik}\, \Phi_{kj} \tag{6}$$

The energy distribution matrix [G] represents the relative magnitude of energy associated with the degrees of freedom. The column sum of matrix [G] is the total energy of the degrees of freedom in the corresponding mode with J^{th} diagonal term of $[\Phi]^T\,[M]\,[\Phi] = [I]$

$$\sum_{i=1}^{N} g_{ij} = \sum_{i=1}^{N} \sum_{k=1}^{N} \Phi_{ij}\, M_{ik}\, \Phi_{kj} = 1 \quad j = 1, \ldots, N \tag{7}$$

Kammer [11] suggested that a partial sum of the energy ranging from 0.4 to 0.5 would be sufficient for the primary set to describe an eigenmode accurately

$$Ec_j = \sum_{i} g_{ij} = \sum_{i} \sum_{k=1}^{N} \Phi_{ij}\, M_{ik}\, \Phi_{kj} \tag{8}$$

In a similar line the row sum can be obtained. i^{th} diagonal term of $[\Phi]^T\,[M]\,[\Phi]^T = [I]$ as

$$\sum_{j=1}^{N} g_{ij} = \sum_{j=1}^{N} \sum_{k=1}^{N} \Phi_{ij}\, M_{ik}\, \Phi_{kj} = 1 \quad i = 1, \ldots, N \tag{9}$$

The row sum of each degree of freedom becomes unity, representing the same degree of importance to the whole Eigen system. For the lowest L eigenmodes the partial row sum is written as

$$Er_j = \sum_{j=1}^{N} g_{ij} = \sum_{j=1}^{N} \sum_{k=1}^{N} \Phi_{ij}\, M_{ik}\, \Phi_{kj} \tag{10}$$

Both the column and the row sums have unit value when all DOF and eigen modes are considered. Hence, the partial column sum represents the contribution of the selected DOF in each mode and can be used to estimate the accuracy of the solution. Where as the partial row sum of a DOF indicate the degree of importance to the modes of interest and can be used as a selection criterion.

Ritz Vectors [5, 6, 14, 15]

Ritz vectors calculations are simple for dynamic problems for the estimation of the energy distribution. The diagonal terms of the mass matrix are used to get the first Ritz vector. For the sake of completeness a brief description of Ritz vector iteration and stiffness and mass matrix normalization process is given in Appendix.

$$[K_{ii}]\left\{X^{(1)}\right\} = [M_{ii}] \tag{11}$$

where $X^{(1)}$ is the first set of Ritz vectors. Additional sets of Ritz vectors are obtained through inverse iteration and ortho-normalisation with respect to the mass matrix.

$$[K]\left\{X^{(k+1)}\right\} = [M]\left\{X^{(k)}\right\} \tag{12}$$

Condensation Technique [12]

As stated earlier, in Guyan condensation, the inertia terms are ignored while constructing the transformation matrix; hence the process is only exact for static problems. For dynamic problems, the accuracy is usually low. Zu-Quing Qu and Panneer Selvam [10] have shown that by a successive iteration process the inertial contributions of the transformation matrix can be improved. In the following sections, the procedure for carrying out both static condensation (Guyan's reduction) and improvement in this process using an iterative process (which happened to be the main theme of the present work) is briefly discussed.

Static Condensation

The dynamic equation of an undamped system is given by

$$[M]\left\{\ddot{U}\right\} + [K]\left\{U\right\} = \left\{F\right\} \tag{13}$$

where $\{F\}$ is an external force vector and $\{U\}$ and $\{\ddot{U}\}$ are the displacement and acceleration response vectors of the structure under the force $\{F\}$. If the total degrees of freedom of the full model are divided into the master and slave degrees of freedom and denoted by m and s, respectively. Eqn. (13) can be rewritten in a partitioned form as

$$\begin{pmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{pmatrix} \begin{Bmatrix} \ddot{U}_m \\ \ddot{U}_s \end{Bmatrix} + \begin{pmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{pmatrix} \begin{Bmatrix} U_m \\ U_s \end{Bmatrix} = \begin{Bmatrix} F_m \\ F_S \end{Bmatrix} \tag{14}$$

As stated earlier, the master DOF can be selected on the basis of either the energy or Ritz vectors criteria as given in the previous sections. In addition, the master DOF should include 1) the boundary DOF in case of a sub-structure analysis, 2) those DOF on which excited forces are located, and 3) those DOF whose displacements are of interest. Based on the selection of these master DOFs, the sub vector $\{F_S\}$ are made equal to zero. Hence, the second equation of Eqn.(14) is reduced to

$$M_{sm}\ddot{U}_m + M_{ss}\ddot{U}_s + K_{sm}U_m + K_{ss}U_s = 0 \tag{15}$$

which leads to

$$U_s = -K_{ss}^{-1}\left(M_{sm}\ddot{U}_m + M_{ss}\ddot{U}_s + K_{sm}U_m\right) \tag{16}$$

Assuming in Eq. (16),

$$\ddot{U}_s = 0 \text{ and } \ddot{U}_m = 0 \tag{17}$$

$$U_s = -K_{ss}^{-1}K_{sm}U_m = R^{(0)}U_m \quad \text{where } R^{(0)} = -K_{ss}^{-1}K_{sm} \tag{18}$$

The relation matrix $R^{(0)}$ defined in Eqn. (18) represents the static transformation matrix. The inertia forces are not considered for obtaining the relation matrix $R^{(0)}$. Hence, it is exact only for static problems. A co-ordinate transformation matrix $T^{(0)}$ can defined as

$$T^{(0)} = \begin{pmatrix} I \\ R^{(0)} \end{pmatrix} \tag{19}$$

The displacement vector $\{U\}$ and acceleration vector $\{\ddot{U}\}$ can be expressed in terms of $T^{(0)}$ as

$$\left\{U\right\} = T^{(0)}\left\{U_m\right\} \text{ and } \ddot{U} = T^{(0)}\left\{\ddot{U}_m\right\} \tag{20}$$

Introducing Eqn. (20) into Eqn. (13) and pre-multiplying both sides of the equation by the transpose of matrix $T^{(0)}$ and on simplification one can get

$$[M_s^{(0)}]\left\{\ddot{U}_m\right\} + [K_s^{(0)}]\left\{U_m\right\} = \left\{F_m\right\} \tag{21}$$

where $K^{s(0)}$ and $M_s^{(0)}$ are stiffness and mass matrices of the static system model given by

$$K_s^{(0)} = (T^{(0)})^T[K]T^{(0)} \text{ and } M_s^{(0)} = (T^{(0)})^T[M]T^{(0)} \tag{22}$$

Iteration Approach to Improve the Relation Matrix for Dynamic Problems

To improve the solution accuracy of dynamic problem using static condensation, the relation matrix $R^{(0)}$ is modified as follows. The free vibration of the system corresponding to Eqn. (21) is

$$M_s^{(0)} \{\ddot{U}_m\} + K_s^{(0)} \{U_m\} = 0 \tag{23}$$

Equation (23) leads to

$$\{\ddot{U}_m\} = -(M_s^{(0)})^{-1} K_s^{(0)} U_m \tag{24}$$

By differentiating both sides of Eqn. (18) with respect to time twice, one obtains

$$\{\ddot{U}_s\} = R^{(0)} \{\ddot{U}_m\} \tag{25}$$

Substituting Eqn. (24) into the right side of Eqn. (25)

$$\ddot{U}_s = -R^{(0)} (M_s^{(0)})^{-1} K_s^{(0)} U_m \tag{26}$$

Using Eqns.(24) and (26), in Eqn.(16), the resulting equation can be written as

$$U_s = K_{ss}^{-1}\left[(M_{sm} + M_{ss} R^{(0)})(M_s^{(0)})^{-1} K_s^{(0)} - K_{sm}\right] U_m \tag{27}$$

According to the definition of the relation matrix R^0 given in Eqn. (18), its first approximation results from Eqn. (27) is written as

$$R^{(1)} = K_{ss}^{-1}\left[(M_{sm} + M_{ss} R^{(0)})(M_s^{(0)})^{-1} K_s^{(0)} - K_{sm}\right] \tag{28}$$

The first order approximation of stiffness matrix $K_s^{(1)}$ and mass matrix $M_s^{(1)}$ of the system can be obtained from Eqs. (19) and (22) by using relation matrix $R^{(1)}$. The accuracy of the matrices $K_s^{(1)}$ and $M_s^{(1)}$ is higher than the matrix $K_s^{(0)}$ and $M_s^{(1)}$ because the inertia terms are improved in relation matrix $R^{(1)}$. Repeating the procedure from Eqs. (19 - 28), (i-1) times, the i^{th} approximation of the relation matrix, stiffness matrix, and mass matrix of system are defined as

$$R^{(i)} = K_{ss}^{-1}\left[(M_{sm} + M_{ss} R^{(i-1)})(M_s^{(i-1)})^{-1} K_s^{(i-1)} - K_{sm}\right] \tag{29}$$

$$K_s^{(i)} = (T^{(i)})^T K T^{(i)} \quad \text{and} \quad M_s^{(i)} = (T^{(i)})^T M T^{(i)}$$

$$\text{with } T^{(i)} = \begin{bmatrix} I \\ R^{(i)} \end{bmatrix}$$

It is shown in Eqs. (18) and (29) that an inversion of matrix K_{ss} is required for a relation matrix. When the number of rigid degrees of freedom of the system is larger than the number of master degrees of freedom, the matrix is singular and cannot be inverted directly. In such a case *pseudo inverse* of the singular K_{ss} matrix can be used which is given in the form $[K_{ss}^T K_{ss}]^{-1} K_{ss}^T$.

In case of analysis by substructure-method, the boundary DOF are part of primary set and the mass contribution of these DOF gets included in the condensed M_s matrix. However, sometimes the degrees of freedom corresponding to rotational coordinates (as in such case of flat plate) have no mass contribution, and as such the M_s matrix is singular. In such a case also pseudo inverse can be carried out and is given as $[M_{ss}^T M_{ss}]^{-1} M_{ss}^T$.

Numerical Examples

Two examples are considered to illustrate the effectiveness of the present approach. First the primary DOF of a given problem are obtained using either energy/or Ritz vector approaches as stated in the previous sections and followed by iterative condensation process given in the previous section using Eqn. (29). The example problems considered are 1) 'L' shaped cantilever beam [1,13] and 2) Dynamic response of a floating raft system [10].

'L' Shaped Cantilever Beam

An 'L' shaped cantilever beam is shown in Fig. 1 with its geometric and material properties. The total degrees of freedom for the beam are 15. The natural frequencies for the beam are shown in Table-1. Degrees of freedom sets (labelled as Case A, Case B, Case C, and Case D) based on Energy method and Ritz vectors for both sequential selection and partial row-sum schemes are given in Table-2. Table-3 gives the energy distribution for each mode corresponding to each DOF. Both for Ritz vectors and

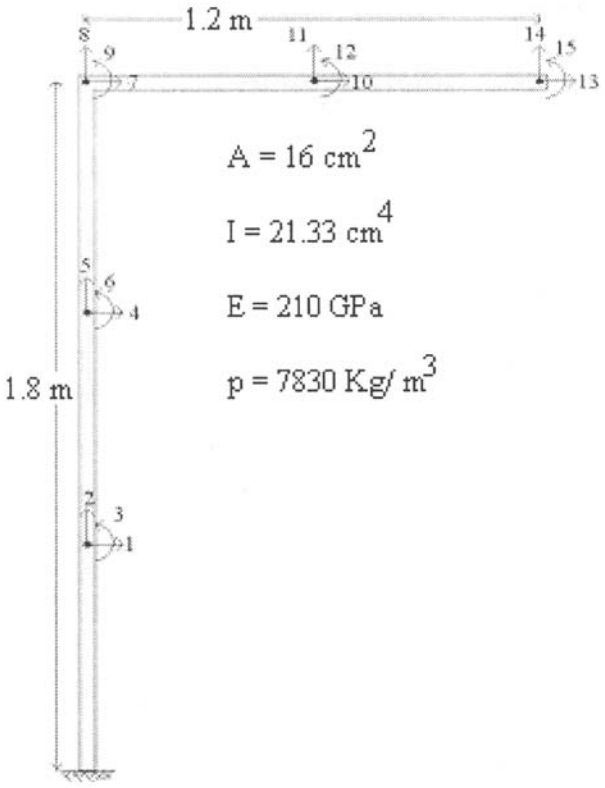

Fig.1 L-Shaped cantilever beam

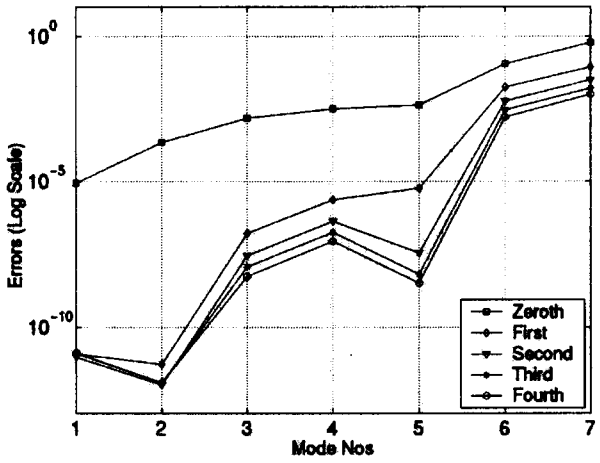

Fig.2 Errors of frequencies for diferent iterations Case-A

eigen vectors happen to be the basis for sequential or row sum selection of DOFs. The errors [$\omega_{reduced}$ - ω_{exact}) / ω_{exact}] of frequencies for these cases from A to D are shown in Figs. 2 to 5 respectively for zeroth, first, second, third and fourth iterations. Ki-Ook and Young [1] obtained the solution by selecting the primary DOF using

Table-1 : Natural frequencies of 'L' shaped beam in rad/sec			
1.	29.88	6.	2245.41
2.	97.31	7.	2853.73
3.	378.06	8.	4257.61
4.	743.35	9.	4631.99
5.	1121.7	10.	6680.66

Table-2 : Primary Degre of fredom for different schemes	
Sequential Selection	
Energy Method (Case A)	10 14 4 11 1 9 12
Ritz Vector (Case B)	10 14 1 4 11 5 9
Row sum Selection	
Energy Method (Case C)	14 1 4 11 12 6 3
Ritz Vector (Case D)	14 11 4 1 9 12 10

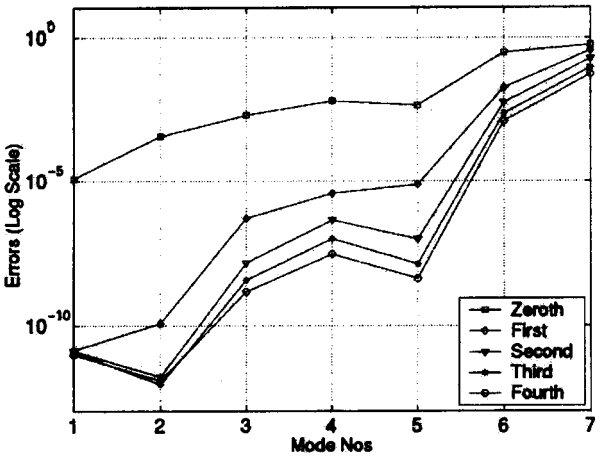

Fig.3 Errors of frequencies for diferent iterations Case-B

sequential selection process where the DOF with largest energy in each mode (exact modes/Ritz vectors) and no attempt is made to improve the frequencies where as focus of the current work is to improve the estimated frequencies using the iterative process. While in the work done by Suarez and Singh [13] uses both Matt's and Shah and Raymund's schemes in the selection of primary DOF for Guyan's reduction. While in the current approach as stated earlier, exact eign modes and Ritz vectors are selected by either sequential process or by row sum process and are further improved by updating the inertial contribution by

Table-3 : Energy Distribution over DOF for each Mode

3A : Ritz Vectors

Ritz Vectors																	
DOF	Mode Nos		3	4	5	6	7	8	9	10	11	12	13	14	15	*Sum	**DOF
1	0.0108	0.0178	0.5323	0.3097	0.0359	0.0187	0.0107	0.0502	-0.002	-0.0003	0.0149	0.0008	0	0	0	0.9359	14
2	0	0	0.0003	0.0072	0.0277	0.0567	0.0127	0	0.0023	0.0298	0.0074	0.1692	0.0325	0.0101	0.6441	0.1046	11
3	0.0021	0.0023	0.0151	0.0365	0.0202	0.0008	-0.002	0.1394	0.4641	0.003	0.3072	0.0106	0.0001	0.0001	0.0003	0.0753	4
4	0.1035	0.0695	0.2289	0.4229	-0.002	0.0889	0.0355	0.003	0.0231	0.0138	0.012	-1E-04	0.0001	0	0.0002	0.9477	1
5	0	0	0.0007	0.0202	0.0774	0.1544	0.0225	0.0079	0.038	0.0101	0.1917	0.3693	0.0094	0.0985	-1E-04	0.2752	9
6	0.004	0.0001	0.0426	0.0288	0.0057	0.0022	-0.001	0.5797	-0.019	0.0023	0.3197	0.0304	0.001	0.0009	0.003	0.0823	12
7	0.2989	0.0236	0.0295	-0.008	0.0017	0.0013	-0.016	-0.002	0.0105	0.0452	0.0207	0.0446	0.2401	0.3048	0.0043	0.3313	10
8	0	0.0001	0.0023	0.0113	0.1408	0.1195	-0.036	0.0149	0.0268	0.146	0.0118	0.2226	0.0014	0.1052	0.2332	0.238	
9	-0.007	0.0116	0.0035	0.077	-0.032	0.0446	0.526	-0.01	0.0052	0.3033	-0.009	0.0489	0.0264	0.0029	0.0084	0.6235	
10	0.3191	0.0186	0.047	0.0462	0.003	0.0004	0.005	0.015	0.0002	0.0265	0.0026	0.0392	0.0291	0.4023	0.0458	0.4393	
11	0.0433	0.2769	0.0034	0.0001	0.4584	0.1199	0.0498	-0.006	0.0368	0.0087	0.0083	0.0003	0	0	0.0001	0.9518	
12	0.0026	0.0269	0.0106	-3E-04	0.1128	0.0299	0.3791	0.0332	-0.032	0.4049	-0.02	0.0197	0.0217	0.0104	0.0003	0.5616	
13	0.1596	0.0093	0.0235	0.0233	0.0015	0.0003	0.0017	0.0126	0.002	0.0273	0.0019	0.0063	0.6156	0.0557	0.0593	0.2192	
14	0.0685	0.5972	0.0639	0.0335	0.0628	0.2742	0.0195	-0.021	-0.063	-0.0054	-0.017	-0.007	-0.005	-0.002	-2E-04	1.1196	
15	-0.005	-0.054	-0.004	-0.009	0.0854	0.0883	-0.008	0.1827	0.5068	-0.0153	0.1465	0.0454	0.027	0.0109	0.0014	0.0945	
	10	14	1	4	11	5	9	6	15	12	3	8	13	7	2		

3B : Eigen Vectors

Eigen Vectors																	
DOF	Mode Nos		3	4	5	6	7	8	9	10	11	12	13	14	15	*Sum	**DOF
1	0.0065	0.0192	0.3476	0.1371	0.4089	0.05	0.0087	0	0.0113	0.01	0.0005	0	0	0	0	0.978	14
2	0	0	0	0	0	0.0003	0.0029	0.1214	0.0053	0.0193	0	0.2308	0.3743	0.2327	0.0128	0.0032	1
3	0.0014	0.0028	0.0257	0.0008	0.0157	0.1782	0.2371	0.0002	0.4182	0.1155	0.0039	0.0002	0.0003	0.0001	0	0.4617	4
4	0.0751	0.0937	0.3747	0.0024	0.3556	0.0066	0.0292	0	0.0608	0.0005	0.0009	0	0.0003	0.0001	0	0.9373	11
5	0	0	0	0.0001	0	0.0012	0.0104	0.3768	0.0158	0.0394	0	0.027	0.0015	0.4847	0.0431	0.0117	12
6	0.0037	0.0005	0.0265	0.0264	0.0243	0.2345	0.1908	0	0.1061	0.3529	0.0282	0.0012	0.0037	0.001	0.0001	0.5067	6
7	0.2639	0.0598	0.0307	0	-0.002	-0.013	-0.002	0	-0.012	0	0.0319	0.2955	0.1899	0.0009	0.1572	0.337	3
8	0	0	0	0.0024	0.0002	0.0119	0.0396	0.3924	0.0007	-0.0182	0	0.1534	0.2001	0.1848	0.0326	0.0541	
9	-0.007	0.0114	0.0151	0.0715	0.0153	0.2441	0.0199	-0.002	0.2261	0.2961	0.0626	0.0043	0.0353	0.0023	0.0051	0.37	
10	0.286	0.0531	0.0646	0	0.0278	0.0094	0.0058	0	0.0121	0.0016	0.0096	0.0137	0.0004	0.0476	0.4685	0.4467	
11	0.0852	0.2341	0.0052	0.4821	0.1019	0.0188	0.0001	0.0096	0.0206	0.0293	0.013	0	0.0001	0	0	0.9274	
12	0.0059	0.024	0.0097	0.028	0.0001	0.1731	0.3402	0.0616	0.072	0.0046	0.2527	0.02	0.007	0.0001	0.001	0.581	
13	0.143	0.0265	0.0323	0	0.0141	0.005	0.0033	0	0.0082	0.0015	0.0257	0.2334	0.1822	0.0457	0.279	0.2242	
14	0.1484	0.5224	0.0824	0.3014	0.0408	0.0391	0.0258	-0.004	-0.007	-0.0291	-0.116	-0.004	-0.001	0	-1E-04	1.1603	
15	-0.012	-0.048	-0.015	-0.052	-0.003	0.0411	0.0879	0.0432	0.0613	0.1767	0.6868	0.0246	0.0058	0.0001	0.0005	0.0009	
	***10	14	4	11	1	9	12	8	3	6	15	7	2	5	13		

*** Row sum of the Energy values for first 7 modes**

**** DOF corresponding to first 7 maximum values of the row sum to be included as a set of Primary DOF set (Row sum selection)**

***** DOF corresponding to maximum values of energy in each mode to be included as a set of Primary DOF (sequential selection)**

iterative process. Figs. 2-5, show that the accuracy of frequencies obtained with the present approach is very high and that with increase in number of iterations the accuracy is further improved. Figs. 2-5 also shows that the errors in frequencies for the lowest modes (lowest 5 in above problem) after first iteration are very low (less than 0.0001%) and thus first iteration is itself sufficient to meet the required acceptable accuracy.

From Tables-2 and 3, it can be seen that the DOF set, selected on the basis of energy method is almost identical to that selected on basis of Ritz vectors. Also, by comparing Figs.2 to 5, it can be seen that the accuracy of frequencies by selecting the primary set based on Ritz vectors is as good as that when the set is selected based on energy method. Thus Ritz vectors, which can be easily calculated compared to eigenvectors (which are not known prior to the solution) can be used for choosing the master DOF. The energy criteria used to select the primary DOF can also provide a guideline on how many DOF should be included for a given problem.

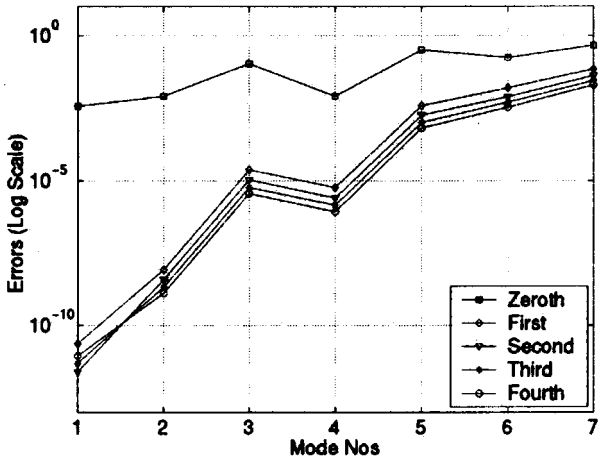

Fig.4 Errors of frequencies for diferent iterations Case-C

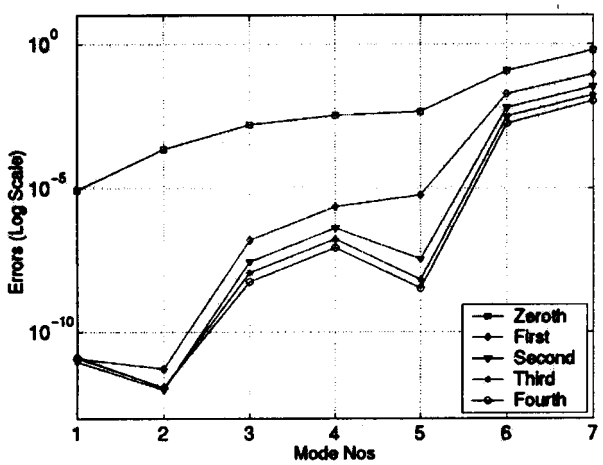

Fig.5 Errors of frequencies for diferent iterations Case-D

Floating-Raft Isolation System

The efficiency of the present approach both for the extraction of modes and for response of a system is shown by a floating- raft isolation system [10]. The floating- raft isolation system will protect the equipment in ships or submarines from damage and let them work normally when the ships or submarines are subjected to strong external loads or shocks. It is a compound dynamic system. It contains springs, dampers, machines to be isolated, a raft frame and a foundation system.

Problem description

A floating-raft isolation system (Fig. 6) is considered here for the illustration of dynamic condensation. For convenience, the damping is not considered for this example. The machines to be isolated are denoted by m_1 = 100 Kg and m_2 = 120 Kg. A and B are rectangle plates and denote the raft frame and base flat frame, respectively. Their lengths, widths and thicknesses are 1.2, 0.8, and 0.02m and 2.8, 0.8, and 0.04 m, respectively. Their modulus of elasticity = 2.0E11 N/m^2 and their mass density = 7800Kg/m^3. Here, k_1 = 1.0E5 N/m and k_2 = 5.0E5 N/m. The two short sides of plate B are simply supported, and the two long sides are free, where as all the four sides of plate A are free.

Finite element model for the floating- raft system

The finite element model of the raft frame (plate A) is shown in Fig.7. The model has 24 rectangle elements, 35

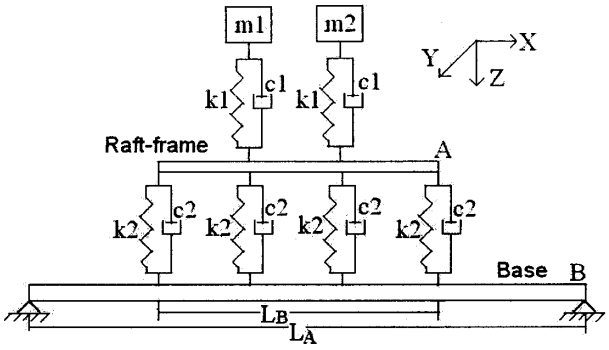

Fig.6 Schematic of a floating-raft isolation system [10]

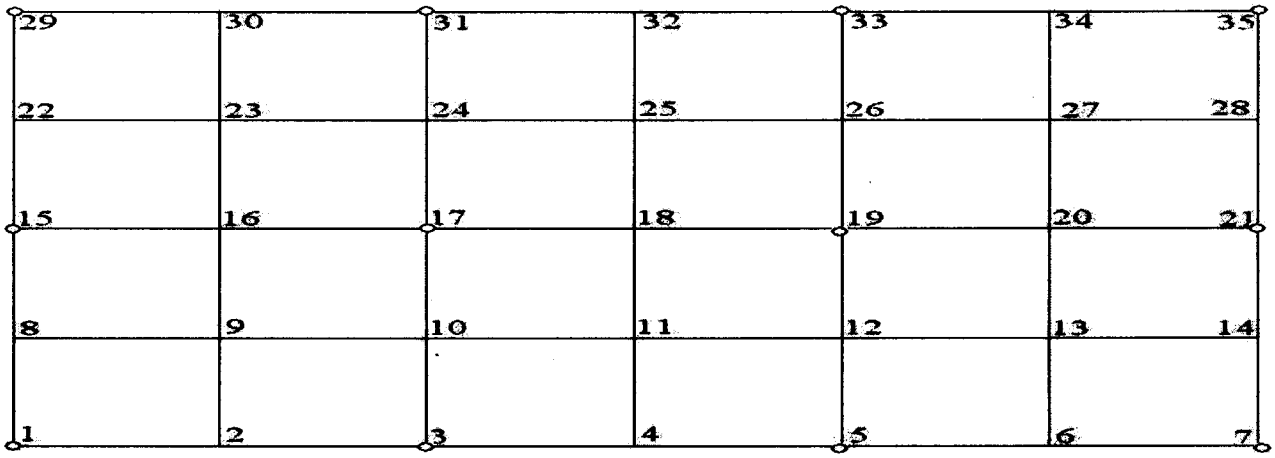

Fig.7 Finite element model of the raft frame (Plate A)

nodes, and 105 DOF (Z-translation, X-rotation and Y-rotation; 3 DOF/node). The nodes that are connected with spring's K_1 are 17 and 19. The nodes connected with springs K_2 are 1, 3, 5, 7, 15, 17, 19, 21, 29, 31, 33, and 35 (shown by circle marks in Fig 7). The finite element model of the base (plate B) is showing in Fig.8. The model has 14 rectangle-elements, 24 nodes, and 66 DOF. The nodes connected with springs K_2 are 3, 4, 5, 6, 11, 12, 13, 14, 19, 20, 21 and 22 (shown by circle marks in Fig. 8).

Application of the condensation technique

Assuming $[K]_A$, $[M]_A$ and $[K]_B$, $[M]_B$ are the stiffness and mass matrices for plate A and B respectively. Energy distributions over the DOF for both plates are calculated. Based on the energy distribution, primary DOF for each plate are selected. In addition the common DOF between the two plates are also retained in the primary DOF set. Table-4 shows the primary DOF set for both the plates.

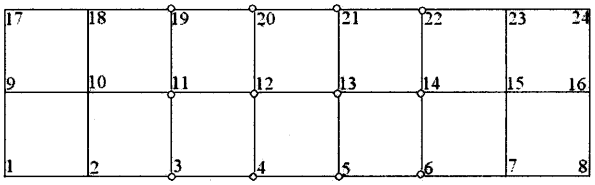

Fig.8 Finite element model of the base (Plate B)

The iteration procedure as explained in the previous section is worked out for each plate. Denoting reduced stiffness and mass matrices by $[K_r]_A$, and $[M_r]_A$ for plate A and by $[K_r]_B$ and $[M_r]_B$ for plate B, and denoting K_c and M_c as stiffness and mass matrices pertaining to the springs and machines, the global stiffness and mass matrices for the floating-raft isolation system is obtained as below.

$$\bar{K} = \begin{bmatrix} [K_r]_A & [0] & 0 & 0 \\ [0] & [K_r]_B & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + K_c \text{ and}$$

$$\bar{M} = \begin{bmatrix} [M_r]_A & [0] & 0 & 0 \\ [0] & [M_r]_B & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + M_c \qquad (30)$$

The global Finite element model of the floating-raft isolation system has 173 DOF. However, the reduced finite element model constructed by using the present approach has 36 DOF, which is much fewer than the global model. The first 10 natural frequencies obtained for this problem in the study based on a) the global model, b) Component Synthesis method (CMS) and c) that from the

Table-4 : Primary DOF set for plates A and B

Plate	Primary DOF set (Z-Translation)																	
A	1	2	3	5	7	8	9	15	17	18	19	21	29	31	32	33	34	35
B	2	3	4	5	6	7	10	11	12	13	14	15	19	20	21	22		

Table-5 : Natural frequencies of the floating-raft isolation system

Mode No.	Natural Frequencies (Hz)			% Error (after 1st iteration) $\left(\frac{\omega_{red} - \omega_{exact}}{\omega_{exact}}\right) * 100$
	Full Model	CMS	Reduced Model	
1.	4.3755	4.3755	4.3760	0.01
2.	4.8916	4.8916	4.8921	0.01
3.	10.8502	10.8497	10.8513	0.01
4.	36.5761	36.5213	36.6492	0.20
5.	36.7144	36.6778	36.7878	0.21
6.	37.7791	37.7734	37.8547	0.25
7.	56.4048	56.1242	56.4499	0.08
8.	66.7382	66.7032	66.8049	0.1
9.	86.1166	85.6711	86.3749	0.30
10.	88.9686	88.4820	89.0220	0.06
Size (DOF)	**173**	**36**	**36**	

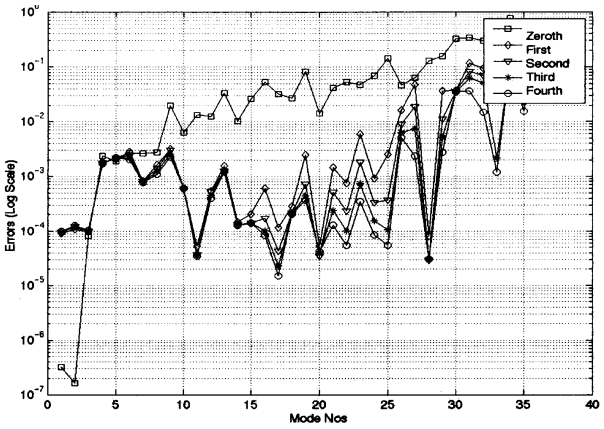

Fig.9 Errors of the frequencies of condensed model for diferent iterations

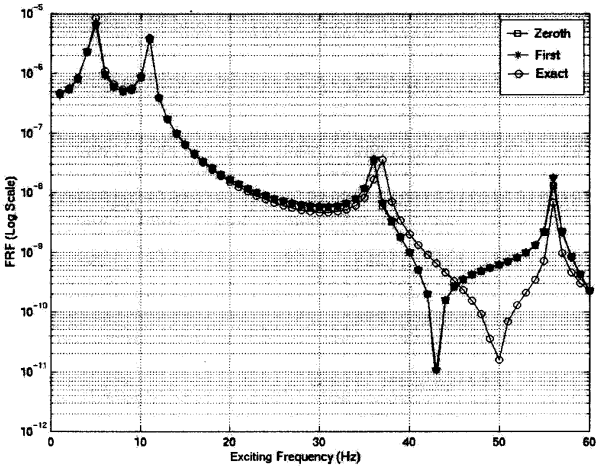

Fig.10 Comparison of FRFs for zeroth and first iteration with exact solution

reduced model. These frequencies are listed in Table-5 along with the percentage errors. The errors of frequencies are shown in Fig.9 for zeroth, first, second, third and fourth iterations.

From Fig. 9 it can be seen that the accuracy of the first three natural frequencies, which are obtained from the reduced finite element model based on the present approach with initial approximation of the system, is very high. The results of Ref. [10] (obtained using 179 DOF for full model and 26 DOF for the reduced model) and the present solutions (apart from full and reduced models CMS approach is used) are identical. However, in both the solutions the errors become larger and larger with increase in frequency. When iteration is applied, the accuracy of frequencies of the reduced model improves. However this does not have much effect on the accuracy of the first three frequencies. Accuracies of the first three frequencies of the reduced model reduced when iterations are adopted. Fortunately, the errors of these frequencies are still less than 0.3% for the first ten modes considered (with first iteration) and have little effect on the accuracy of response.

Frequency response of the floating-raft system

The frequency response function for the raft is calculated. Fig.10 shows the frequency response functions (FRFs) for the exact model and for reduced model with zeroth and first iteration. The excited force is located on m_1 in the Z direction and the response degree of freedom is the displacement at node 12 (plate B) in the same direction. For convenience, only the absolute values of FRFs are shown in Fig.10. The results show that the accuracy of the reduced model can be efficiently improved by use of the current approach.

Conclusions

An approach to improve the dynamic results using Guyan's static condensation process by improving the mass contribution in the condensation process by an iterative approach is illustrated. Further, the initial vector (primary DOF) is selected using energy method or Ritz vectors. The effectiveness of the present method is validated with two examples 1) An 'L' shaped beam and 2) Dynamic response of a floating-raft isolation system (involving the application of the same procedure for substructures). The proposed reduced model can be applied to dynamic analysis both in time and frequency domains, passive and active vibration control, test-analysis model correlation etc. The present approach can also be applied for large scale problem where sub-structuring technique plays a critical role.

References

1. Ki-Ook Kim and Young-Jae Choi., "Energy Method for Selection of Degrees of Freedom in Condensation", AIAA Journal, 38(7), 1253-1259,2000.

2. Mac Neal Schwendler Corp., MSC/NASTRAN Handbook for Dynamic Analysis, Los Angles, 4.1-4.5,1983.

3. Bathe K. J., Finite Element Procedures, Prentice - Hall, Upper Saddle River, NJ, 844, 1996.

4. Kim K. O., Anderson W.J. and Sandstorm R.E., "Non-Linear Inverse Perturbation Method in Dynamic Analysis", AIAA Journal, 21(9),1310-1316, 1983.

5. Wilson E.L., "A New Method of Dynamic Analysis for Linear and Nonlinear Systems", Finite Elements in Analysis and Design, 1, 21-23,1985.

6. Arnold R.R., Citerley R.L., Chargin M. and Galant D., "Application of Ritz Vectors for Dynamic Analysis of Large Structures", Computers and Structures, 21(5), 901-907,1985.

7. Thomas, D.L., "Errors in Natural Frequency Calculations using Eigen Value Economisation", International Journal of Numerical Methods in Engineering, 18, 1521-1527,1982.

8. Kim, K.O., "Perturbation Method Condensation of Eigen Problems", AIAA Journal, 36(9), 1757-1760,1998.

9. Coppolino R.N., "Automated Response DOF Selection for Mapping of Experimental Normal Modes", Proceedings of 16th International Modal Analysis Conference, Society for Experimental Mechanics, Santa Barbara, CA, 70-76,1998.

10. Zu-Quing Qu and Panneer Selvam, R., "Dynamic Superelement Modelling Method for Compound Dynamic Systems", AIAA Journal, 38(6), 1078-1083,2000.

11. Kamme,r D.C., "Sensor Placement For On-Orbit Modal Identifications and Correlation of Large Space Structures", Journal Of Guidance, Control and Dynamics, 14(2),251-259,1991.

12. Guyan R.L., "Reduction of Stiffness and Mass Matrices", AIAA Journal, 3(2), 380,1965.

13. Suarez, L. E. and Singh M.P., "Dynamic Condensation Method for Structural Eigen Value Analysis," AIAA Journal, 30(4), 1046-1054, 1992.

14. Wilson, E.L. et al., "Dynamic Analysis by Direct Superposition of Ritz vectors," Earthquake Engineering Structural Dynamics", Vol. 10,813-823, 1982.

15. Wilson E. L. and Iton,T., "An Eigen Solution Strategy for Large Systems", Journal of Computers and Structures, Vol.16, No.1-4, 259-265,1983.

Appendix

Ritz Vectors and Normalization of Mass and Stiffness Matrices

The major reason to compute eigen modes and mode shapes (eigen values and eigen vectors) is that they are used to uncouple the dynamic equations for sub-sequent superposition and or for response analysis. The numerical effort required to calculate the exact eigen solution can be enormous for a structural system if a large number of modes are required. With the development of high speed computing systems the use of exact eigen vectors replaced by the use of Ritz vectors as basis for large response analysis. The Ritz vectors give accurate results with less computational efforts than that of exact eigen vectors. In this appendix, the analytical procedure for the generation of Ritz vectors and their normalization process used with mass and stiffness is presented.

Generation of Ritz Vectors

Initial Vector Calculation U_1

- Triangularisation of stiffness Matrix. $K = L^t \mathbf{D} L$
- Solve for the initial set of static displacement vectors resulting from spatial load patterns **F** in the form $F\ U_s = \mathbf{K}$.
- Make this set of vectors $\mathbf{U_s}$, mass or stiffness orthogonal as required.

Generation of sets of Ritz vectors i = 2,.....N

- Make the set of vectors, $\mathbf{X_i}$, stiffness and mass orthogonal, $\underline{U}_i$.
- Solve for sets of vectors, $\mathbf{X_i}$, using the recursive relationship such that $\mathbf{K\ X_i = MU_{i-1}}$.
- Use Modified Gram-Schmidt method to make $\mathbf{U_i}$ orthogonal to all previously calculated vectors and normalized so that $U_i^t\ \mathbf{MU_i = I}$.

Making the vectors stiffness orthogonal

- The N x N eigen value problem $[\overline{K} - \omega^2 I]\ Z = 0$ is solved with $\overline{K} = \mathbf{U_i^t\ K\ U_i}$.
- Compute the stiffness orthogonal Ritz vectors, $\varphi = UZ$.

A Physical Explanation

The physical explanation for the method is the recognition that the dynamic response of a structure will be a function of the spatial load distribution. For the case of un-damped, dynamic equilibrium equations of an elastic structure can be written in the following form

$$[M]\ \{\ddot{U}(t)\} + [K]\ \{U(t)\} = \{R(t)\} \qquad \text{(A-1)}$$

where R(t), the time dependent loading acting on the structure which can be separated into a spatial load vector F and a time function G(t) in the form

$$R(t) = \sum_{i=1}^{n} f_i\, g_i(t) = F\, G(t) \qquad \text{(A-2)}$$

The time function G(t) can always be expanded into a Fourier series of sine and cosine functions. Hence, neglecting damping, a typical dynamic equilibrium equation to be solved as shown in Eqn. (A-3)

$$[M]\ \{\ddot{U}(t)\} + [K]\ \{U(t)\} = \{F \sin \omega(t)\} \qquad \text{(A-3)}$$

Hence, the exact dynamic response for a typical loading frequency is of the following form

$$KU = F + \omega^2\, MU \qquad \text{(A-4)}$$

This equation cannot be solved directly because of the unknown frequency of the loading. However, a series of stiffness and mass orthogonal vectors can be calculated that will satisfy this equation using a perturbation algorithm. The first set of vectors is a static deformation caused by the spatial distribution of the dynamic load vector F such that

$$K\ U_s = F \qquad \text{(A-5)}$$

From Equation (A-5) it is apparent that the error in the solution by neglecting the inertia forces, can be approximated by

$$F_1 = \omega^2\, M\, U_s \qquad \text{(A-6)}$$

Therefore, an additional set of displacement vectors $U_{\underline{1}}$ can be calculated from

$$KU_1 = F_1 \quad \text{(A-7)}$$

This first Ritz vector is mass-normalized as $U_1 = \overline{\mathbf{U}}_1 / [\overline{\mathbf{U}}_1^{\,t} M \overline{\mathbf{U}}_1]^{1/2}$

The subsequent Ritz vectors are recursively generated and the following recurrence relationship used to compute the next Ritz vector $\mathbf{U_i}$ as

$$K \mathbf{U_i} = \mathbf{M U_{i-1}} \quad \text{(A-8)}$$

This is similar to the one given by Eqn. (12). The linear independence of these Ritz vectors is achieved using the Gram-Schmidt orthogonalisation (mass-orthogonalised with respect to all the previous Ritz vectors followed by mass normalisation of the current Ritz vector). A large number of vectors can be generated by Eqn. (A-8). It is interesting to note that the recursive equation, used to generate the Ritz vectors, is similar to the Lanczos algorithm for calculating exact eigen values and vectors, except that the starting vectors are the static displacements caused by the spatial load distributions.