BENDING VIBRATION AND BUCKLING ANALYSIS OF NONHOMOGENEOUS NANOTUBES USING NONLOCAL ELASTICITY THEORY AND GDQ METHOD

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Abstract

In this paper structural analysis of nonhomogeneous nanotubes has been carried out using nonlocal elasticity theory. Governing differential equations of nonhomogeneous nanotubes are derived. Nonlocal theory of elasticity has been employed to include the scale effect of the nanotubes. Nonlocal parameter, elastic modulus, density and diameter of the cross sections are assumed to be functions of spatial coordinates. General Differential Quadrature (GDQ) method has been employed to solve the governing differential equations of the nanotubes. Various boundary conditions have been applied to the nanotubes. Present results considering nonlocal theory are in good agreement with the results available in the literature. Effect of various geometrical and material parameters on the structural response of the nonhomogeneous nanotubes has been investigated. Present results of the nonhomogeneous nanotubes are useful in the design of the nanotubes.

Keywords: Nanotubes, Differential Quadrature Method, Nonhomogeneous, Bending, Vibration, Buckling

Introduction

Nano sized tubes hold an important area of research for the future structural developments and design in modern aerospace engineering. This is due to their novel mechanical and electronic properties. These nano-tubes have got highly promising applications in nanotube-reinforced ultra-strong composites, MEMS/NEMS devices and smart structures. Analysis of nano tubes are useful in the design of advanced aerospace nano structures. Since the discovery of carbon nanotubes (CNT) by Iijima [1] good amount of research work has been reported in the literature. Review work related to behavior of the CNTs due to novel electronic and mechanical properties are reported by Thostenson et al. [2] and Ronald et al. [3]. Conducting experiments with nanoscale size specimens is found to be difficult and costly. Therefore, development of appropriate mathematical models for CNTs became an important issue. Generally, three approaches have been developed to model CNTs. These approaches are (a) atomistic (b) hybrid atomistic-continuum mechanics and (c) continuum mechanics. Atomistic approach uses (i) classical molecular dynamics simulation, (ii) tight binding molecular dynamics and (iii) density functional models. These models are discussed by Ball [4] and Baughman et al. [5]. But these approaches are computationally intensive and very expensive. So hybrid atomistic-continuum mechanics approach was tried by Bodily and Sun [6] and Li and Chou [7, 8]. In this hybrid approach the CNTs are represented by structural elements. The strain energy is considered to be equivalent of the steric energy. This hybrid approach is computationally less expensive than the atomistic approach. Some researchers employed continuum mechanics approach for the analysis of CNTs. Here single wall carbon nanotubes are modeled by a continuum beam or cylindrical shell elements. This continuum mechanics approach is ideal in analyzing large scale systems containing CNTs. For multi walled CNTs a multi beam model has been proposed by Yoon et al. [9,10]. For more accurate analysis shear deformation theories of beam have been proposed by Wang et al [11], Wang and Vardan [12] and Aydogdu [13]. Eringen [14, 15] developed nonlocal elasticity theory. In this nonlocal elasticity theory scale effect is included. While classical elasticity theory is indifferent to scale effects. Peddieson et al. [16] proposed analysis of nanostructures based on Eringens nonlocal elasticity theory. The nonlocal elasticity theory

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has been further applied to the static and dynamic analysis of single walled and multi walled CNTs by Wang et al. [17], Wang and Varadan [18] and Pin et al. [19].

In all the above mentioned work, analysis of the homogeneous nanotubes (CNT) has been carried out. Seeman [20] found that nonhomogeneous nanotubes are frequently encountered in DNA nanotechnology applications. He observed that different proteins are chemically glued to form nano-architectures of the nanotubes. Rothemund et al. [21] reported that DNA nanotubes are similar in size and shape as carbon nanotubes. They suggested that DNA nanotubes could be easily modified and connected to other structures. Important applications of DNA nanotubes include nano-wires and nano-pipes. Nonhomogeneous nanotubes can be addressed similar to CNTs. To the authors best knowledge no work has been addressed for the analysis of nonhomogeneous nanotubes employing continuum mechanics approach. Therefore in the present work bending, vibration and buckling analyses of a nonhomogeneous nanotube have been carried out and results are discussed.

Formulation

Nonhomogeneous Nanotube

Non-homogeneity imparts additional complexity to the analysis of the nanotubes in the following ways. The material properties viz elastic modulus, nonlocal parameter and density are functions of spatial coordinates. For the beam structure these variations are assumed be in the axial direction. The internal characteristic lengths are different for different materials. Thus variation of nonlocal parameter along the nanotube axial direction needs to be considered. Similarly, different bond lengths will result in variation of the nanotube diameter. Therefore in the analysis of nonhomogeneous nanotubes variation of elastic modulus, density, nonlocal parameter and nanotube diameter along the axial direction are to be included. In the present study a nonhomogeneous nanotube is modeled by nonlocal elastic continuum Euler-Bernoulli beam of annular crossection. Elastic modulus, density, scale factor and diameter of nanotube are assumed to be functions of axial coordinate. For modeling the double walled nonhomogeneous nanotube multi Euler-Bernoulli beam has been employed.

According to Eringen [14] the nonlocal constitutive behavior of a Hookean solid can be represented by the following differential constitutive relation

$$(1-\tau^2 l^2 \nabla^2)\,\sigma = \boldsymbol{t},\ \tau = \frac{\boldsymbol{e_0 a}}{l} \tag{1}$$

Here e_0 is a material constant, 'a' and l are external and internal characteristic lengths respectively. $\boldsymbol{t}$ is the macroscopic stress at a point which is related to strain by generalized Hookes law:

$$t(x) = C(x) : \varepsilon(x) \tag{2}$$

where C is the fourth order elasticity tensor and ':' denotes the double dot product. It is assumed that nonlocal behavior is significant in axial direction of the nanotube. Thus, nonlocal constitutive relation mentioned in equation (1) takes the following form for an isotropic Euler-Bernoulli beam.

$$\sigma(x) - \mu \frac{d^2\sigma(x)}{dx^2} = E\varepsilon(x) \tag{3}$$

Here, $\mu = (e_0 a)^2$ is the scale factor. E is the elastic modulus. As it is a differential relation, non-homogeneity can be incorporated in this equation. For nonhomogeneous case this differential relation (3) takes the following form

$$\sigma(x) - \mu(x)\frac{d^2\sigma}{dx^2} = E(x)\ \varepsilon(x) \tag{4}$$

From the definition of resulting bending moment and strain displacement relation in Euler-Bernoulli beam

$$M = \int_A z\,\sigma\,dA \tag{5}$$

$$\varepsilon = -z\frac{\partial^2 w}{\partial x^2} \tag{6}$$

Using equation (4), (5) and (6) we get following moment-displacement relation

$$M(x) - \mu(x)\frac{d^2 M(x)}{dx^2} = E(x)\,I(x)\frac{d^2 w}{dx^2} \tag{7}$$

Bending Analysis of Single Walled Nanotube (SWNT)

For an Euler-Bernoulli beam acted by distributed load q(x), the equilibrium equation is expressed as

$$q(x) = \frac{d^2 M(x)}{dx^2} \tag{8}$$

Differentiating twice equation (7) and substituting from equation (8), we get the following governing equation for bending of nonhomogeneous beam

$$\Omega_5^{bn} \frac{d^4 w}{dx^4} + \Omega_4^{bn} \frac{d^3 w}{dx^3} + \Omega_3^{bn} \frac{d^2 w}{dx^2} + \Omega_0^{bn} = 0 \tag{9}$$

Ω_0^{bn}, Ω_3^{bn}, Ω_4^{bn}, and Ω_5^{bn} are defined in Appendix.

Vibration Analysis SWNT

For free vibration we have the following equation for equilibrium

$$\frac{\partial^2 M}{\partial x^2} = \rho(x) A(x) \frac{\partial^2 w}{\partial t^2} \tag{10}$$

$\rho(x)$ and $A(x)$ denote density of the material and area of cross section, respectively. Differentiating twice equation (7) and substituting in equation (10), we get the following governing equation for vibration of nonhomogeneous beam

$$\gamma_5^{vb} \frac{\partial^4 w}{\partial x^4} + \gamma_4^{vb} \frac{\partial^3 w}{\partial x^3} + \gamma_3^{vb} \frac{\partial^2 w}{\partial x^2} + \Theta_5^{vb} \frac{\partial^4 w}{\partial x^2 \partial t^2} + \Theta_4^{vb} \frac{\partial^3 w}{\partial x \partial t^2} + \Theta_3^{vb} \frac{\partial^2 w}{\partial t^2} = 0 \tag{11}$$

γ_3^{vb}, γ_4^{vb}, γ_5^{vb}, Θ_3^{vb}, Θ_4^{vb} and Θ_5^{vb} are defined in the Appendix. The above equation is converted to an eigenvalue problem by assuming the periodic function

$$w(x, t) = w(x) e^{i \omega t} \tag{12}$$

Substituting equation (12) into equation (11), we have

$$\Omega_5^{vb} \frac{d^4 w}{dx^4} + \Omega_4^{vb} \frac{d^3 w}{dx^3} + \Omega_3^{vb} \frac{d^2 w}{dx^2} + \Omega_2^{vb} \frac{dw}{dx} + \Omega_1^{vb} w = 0 \tag{13}$$

Ω_1^{vb}, Ω_2^{vb}, Ω_3^{vb}, Ω_4^{vb} and Ω_5^{vb} are defined in the Appendix.

Buckling Analysis SWNT

The equilibrium equation for buckling of an Euler-Bernoulli beam under axial compressive load P is given by

$$\frac{d^2 M}{dx^2} = P \frac{d^2 w}{dx^2} \tag{14}$$

Differentiating twice equation (7) and substituting from equation (14) we get governing equation for buckling of nonhomogeneous nanotube

$$\Omega_5^{bk} \frac{d^4 w}{dx^4} + \Omega_4^{bk} \frac{d^3 w}{dx^3} + \Omega_3^{bk} \frac{d^2 w}{dx^2} = 0 \tag{15}$$

Ω_3^{bk}, Ω_4^{bk} and Ω_5^{bk} are defined in the Appendix.

Boundary Conditions

All four classical boundary conditions have been considered in the analysis. These are (i) Simply supported - Simply supported (S-S) (ii) Clamped Free (C-F) (iii) Clamped Simply supported (C-S) and (iv) Clamped - Clamped (C-C). As we are solving fourth order differential equations (equations 9, 13, 15) on w we need four boundary conditions in w in each case. The displacement and stress boundary conditions associated with S-S, C-F, C-S and C-C cases are listed in Table-1.

Here L denotes the length of the beam. To get four boundary conditions on w we must convert the stress boundary conditions to corresponding displacement boundary conditions. However for C-C cases we have four inherent displacement boundary conditions and no stress boundary conditions, so no need of any conversion. The conversion from stress to displacement boundary conditions must be done by suitable stress-strain relationship given by nonlocal elasticity theory. We can use the moment-displacement relationship of equation (7) to get these nonlocal boundary conditions.

Conversion of $M(x)\big|_{x=0} = 0$ for S-S boundary condition :

Using equations (7) and (8) and putting $x = 0$,

$$\left.\frac{d^2 w}{dx^2}\right|_{x=0} = -\frac{\mu(0)\, q(0)}{E(0)\, I(0)} \tag{16}$$

Table-1 : Displacement and Stress Boundary Conditions for Various Classical end Conditions

End Boundary Condition	Displacement Boundary Condition	Stress Boundary Condition
S-S	$w(x)\|_{x=0} = 0,\ \ w(x)\|_{x=L} = 0$	$M(x)\|_{x=0} = 0,\ \ M(x)\|_{x=L} = 0$
C-F	$w(x)\|_{x=0} = 0,\ \ \frac{dw(x)}{dx}\Big\|_{x=0} = 0$	$M(x)\|_{x=L} = 0,\ \ \frac{dM(x)}{dx}\Big\|_{x=L} = 0$
C-S	$w(x)\|_{x=0} = 0,\ \ \frac{dw(x)}{dx}\Big\|_{x=0} = 0,$ $w(x)\|_{x=L} = 0$	$M(x)\|_{x=L} = 0$
C-C	$w(x)\|_{x=0} = 0,\ \ \frac{dw(x)}{dx}\Big\|_{x=0} = 0,$ $w(x)\|_{x=L} = 0,\ \ \frac{dw(x)}{dx}\Big\|_{x=L} = 0$	- -

Conversion of $M(x)\,|_{x=L} = 0$ for S-S, C-F, C-S boundary condition :

Using equations (7) and (8) and putting $x = L$,

$$\frac{d^2w}{dx^2}\bigg|_{x=L} = -\frac{\mu(L)\,q(L)}{E(L)\,I(L)} \tag{17}$$

Conversion of $\frac{d\,M(x)}{dx}\Big|_{x=L} = 0$ for C-F boundary condition :

Differentiating equation (7), substituting in equation (8) and putting x = L we obtain,

$$\frac{d^3w}{dx^3}\bigg|_{x=L} = -\frac{1}{E(L)\,I(L)}\,\frac{d(\mu(x)\,q(x))}{dx}\bigg|_{x=L}$$

$$-\,\mu(L)\,q(L)\,\frac{d(1/E(x)\,I(x))}{dx}\bigg|_{x=L} \tag{18}$$

Thus we get four boundary conditions on w for all the boundary conditions considered in the present analysis.

GDQ Method

The governing equations for bending, vibration and buckling of nonhomogeneous SWNT are presented in equations (9), (13) and (15), respectively. These equations have been solved by the differential quadrature method (DQM) as introduced by Bellman et al. [22]. The DQ method has been proved to be an efficient numerical technique for the solution of initial and boundary value problems. Bert et al. [23] first employed this method to solve structural mechanics problems. This method has also been applied successfully to a variety of structural problems by Bert and Malik [24] and Shu [25]. The fundamental concept of DQ method is to approximate the partial derivative of a function with respect to a space variable at a grid point by the weighted linear sum of the function values at all grid points in the whole domain. In the present case the computational domain for the problem is $0 \le x \le L$. So we have

$$\frac{d^n g}{d^n}\bigg|_{x=L} = \sum_{j=1}^{N} A_{ij}^{n}\, g(x_j) \tag{19}$$

N is the number of grid points. g is the function to be approximated. A_{ij}^{n} are DQ weighting coefficients which can be calculated from the coordinates of the grid points as follows

$$A_{ij}^{l} = \frac{M(x_i)}{(x_i - x_j)\,M(x_j)},\ \text{for}\ \ i \ne j \tag{20}$$

$$A_{ii}^{l} = -\sum_{j=1,\,j\ne i}^{N} A_{ij}^{l} \tag{21}$$

$$A_{ij}^{m} = m\left(A_{ii}^{m-1} - \frac{A_{ij}^{m-1}}{x_i - x_j}\right),\ \ \text{for}\ \ i \ne j \tag{22}$$

$$A_{ii}^{m} = -\sum_{j=1,\, j\neq i}^{N} A_{ij}^{m} \tag{23}$$

for i, j = 1, 2, 3 ..... , N
m = 2, 3, 4 ..... , N -1

Here

$$M(x_i) = \prod_{k=1,\, k\neq i}^{N} (x_i - x_k) \tag{24}$$

To choose grid point distribution the well accepted mesh governed by the following rule for calculating interpolation points has been adopted

$$x_i = \frac{1}{2}\left[1 - \cos\left(\frac{\pi\, i}{n}\right)\right], \quad i = 0, ..., n \tag{25}$$

By applying DQ rule to equations (9), (13) and (15) we obtain following discretized formulation for equations (9), (13) and (15), respectively.

$$\Omega_5^{bn}(X_i)\sum_{k=1}^{N} A_{ik}^{4} w_i + \Omega_4^{bn}(X_i)\sum_{k=1}^{N} A_{ik}^{3} w_i$$

$$+ \Omega_3^{bn}(X_i)\sum_{k=1}^{N} A_{ik}^{2} w_i + \Omega_0^{bn}(X_i) = 0 \tag{26}$$

$$\Omega_5^{vb}(X_i)\sum_{k=1}^{N} A_{ik}^{4} w_i + \Omega_4^{vb}(X_i)\sum_{k=1}^{N} A_{ik}^{3} w_i = \Omega_3^{vb}(X_i)\sum_{k=1}^{N} A_{ik}^{2} w_i$$

$$\Omega_2^{vb}(X_i)\sum_{k=1}^{N} A_{ik}^{1} w_i + \Omega_1^{vb}(X_i)\, w_i = 0 \tag{27}$$

$$\Omega_5^{bk}(X_i)\sum_{k=1}^{N} A_{ik}^{4} w_i + \Omega_4^{bk}(X_i)\sum_{k=1}^{N} A_{ik}^{3} w_i + \Omega_3^{bk}(X_i)\sum_{k=1}^{N} A_{ik}^{2} w_i = 0 \tag{28}$$

$\Omega_5^{bn}(X_i)$, $\Omega_4^{bn}(X_i)$, $\Omega_5^{vb}(X_i)$, $\Omega_5^{bk}(X_i)$ etc. denote values of Ω_5^{bn}, Ω_4^{bn}, Ω_5^{vb}, Ω_5^{bk} … at the grid coordinate (X_i). These contain first or higher order derivatives of elastic modulus, scale coefficient etc. which can be computed numerically by applying DQ approximation for derivatives. It should be noted that the discretized eigen-value equations (27) and (28) have been reduced to general eigen-value problems. The GDQM technique presented by Shu [25] could efficiently implement the four classical boundary conditions. The systematic algorithm adopted here for implementing the boundary conditions for bending analysis is as follows

Step 1 : Apply discretized governing equations at internal (N-4) grid points (leaving two leftmost and two rightmost grid points) only.

Step 2 : Apply discretized boundary conditions on leftmost and rightmost grid points. This will give four equations corresponding to four boundary conditions.

Step 3 : Express displacements at two leftmost and two rightmost grid points in terms of other displacements at internal (N-4) grid points, using the four equations obtained in step 2.

Step 4 : Substitute for these four displacements of step 3 in the discretized equations obtained in step 1.

Step 5 : Solve these (N-4) equations of step 4 to compute displacements at internal (N-4) grid points.

Step 6 : Update the displacements of two leftmost and two rightmost grid points using the computed displacements in step 5 and expressions obtained in step 3.

For vibration and buckling analyses one needs to solve the general eigen-value problem arising in Step 4.

Results and Discussions

Convergence study of differential quadrature method with various grid points is conducted and results are shown in Fig.1. In this particular example bending analysis has been carried out. Cubic variation of elastic modulus, nonlocal parameter, density and diameter of the nanotube are considered in this convergence study. From Fig.1, one could observe that ten grid points are good enough for reasonably accurate results. Also it was observed that fifteen grid points are enough to achieve reasonably accurate vibration and buckling results. So in all the following computations fifteen grid points are employed. This also reveals the efficiency of DQ method in analyzing nonhomogeneous nanotubes.

At first, bending, vibration and buckling results are obtained by employing local elasticity theory. These re-

Fig.1 Convergence Study of Differential Quadrature Method with Various Grid Points

sults are compared with those obtained by Yang [26] employing distributed transfer function method (DTFM) for all four boundary conditions. After this validation, bending, vibration and buckling results are obtained employing nonlocal elasticity theory. These results are also compared with corresponding results available in literature. Further, nonhomogeneous solutions with nonlocal elasticity theory are obtained. Various variations of elastic modulus, nonlocal parameter, density and diameter of nanotube along the axial direction are included in the investigation.

Validation of Beam Results with Local Theory

Beam with following parameters is considered for the analysis. E = 1 N/m^2, L = 1 m, A = 1 m^2, q = 1 N/m, ρ = 1 kg/m^3 and I = 1 m^4. Maximum deflections, natural frequencies and critical buckling loads have been normalized as follows:

$$\hat{w} = w \times \frac{EI}{qL^4}, \ \hat{f} = f \times L^2 \sqrt{\frac{\rho A}{EI}} \text{ and } \hat{P}_{cr} \times \frac{L^2}{EI} \qquad (29)$$

Table-2 : Non-Dimensional Deflection ($\hat{w}$) of Beams

Boundary Condition	Non-Dimensional max. Deflection ($\hat{w}$) (Yang [26])	Non-Dimensional max. Deflection ($\hat{w}$) (Present)
S-S	0.0130	0.0130
C-F	0.1240	0.1249
C-S	0.0054	0.0054
C-C	0.0026	0.0026

Employing the classical local elasticity theory bending, vibration and buckling results are obtained. The non-dimensional maximum deflection ($\hat{w}$), natural frequencies ($\hat{f}$) and critical buckling loads ($\hat{P}_{cr}$) are listed in Tables-2 to 4, respectively. These results are compared with DTFM (Yang [26]) results. It is observed that present DQM results are in good agreement with results obtained employing DTFM.

Validation of Beam Results with Nonlocal Theory

Reddy [27] obtained bending, vibration and buckling solutions of simply supported beams with nonlocal elasticity theory. Peddieson et al. [16] obtained bending solu-

Table-3 : Non-Dimensional Natural Frequencies ($\hat{f}$) of Beam

Boundary Condition	Mode No.	Non-Dimensional Natural Freq.($\hat{f}$) (Yang [26])	Non-Dimensional Natural Freq.($\hat{f}$) (Present)
S-S	1	9.8690	9.8696
	2	39.5570	39.4684
	3	89.0040	88.8249
C-F	1	3.5231	3.5160
	2	22.0340	22.0345
	3	61.8208	61.6999
C-S	1	15.4490	15.4182
	2	50.0650	49.9648
	3	104.4560	104.2471
C-C	1	22.3730	22.3733
	2	61.7960	61.6728
	3	121.1450	120.9021

Table-4 : Non-Dimensional Critical Buckling Loads ($\hat{P}_{cr}$) of Beam

Boundary Condition	Non-DimensionalCritical Buckling Load ($\hat{P}_{cr}$) (Yang [26])	Non-DimensionalCritical Buckling Load ($\hat{P}_{cr}$) (Present)
S-S	9.8696	9.86960
C-F	2.4670	2.47490
C-S	20.1900	20.1907
C-C	39.4800	39.4784

tion for cantilever beam employing nonlocal elasticity theory. Wang et al. [28] employed nonlocal elasticity theory and obtained buckling solutions for columns with S-S, C-F and C-C boundary conditions. Murmu and Pradhan [29] studied the effect of nonlocal parameter on the response of carbon nanotubes embedded in an elastic medium based on nonlocal continuum mechanics. All these above mentioned solutions reported by Reddy [26], Peddieson et al. [16] and Wang et al. [28] are analytical in nature. But in the present analysis DQ method is employed because of complexity of the governing differential equations for nonhomogeneous naotubes. Material properties and geometrical dimensions of the beams are assumed to be same as mentioned by Reddy [27], Peddieson et al. [16]

Table-5 : Non-Dimensional Deflections ($\hat{w}$) of the Beam with Non-Nonlocal Theory of Elasticity

Nonlocal Parameter (μ)	Boundary Condition	Non-Dimensional max. Deflection ($\hat{w}$) (Reddy [27])	Non-Dimensional max. Deflection ($\hat{w}$) (Peddieson et al. [16])	Non-Dimensional max. Deflection ($\hat{w}$) (Present)
0.5	S-S	0.0756	----	0.0755
	C-F	----	0.1250	0.1249
	C-S	----	----	0.0237
	C-C	----	----	0.0026
1.0	S-S	0.1382	----	0.1380
	C-F	----	0.3750	0.3749
	C-S	----	----	0.0422
	C-C	----	----	0.0026

Table-6 : Non-Dimensional Natural Frequencies ($\hat{f}$) of the Beam with Non-Nonlocal Theory of Elasticity

Nonlocal Parameter	Boundary Condition	Mode No.	Non-Dimensional Natural Frequency ($\hat{f}$) (Reddy [27])	Non-Dimensional Natural Frequency ($\hat{f}$) (Present)
0.5	S-S	1	4.0489	4.0513
		2	8.6643	8.6689
		3	13.1743	13.1809
	C-F	1	----	2.2498
		2	----	11.9413
		3	----	49.8439
	C-S	1	----	5.8851
		2	----	10.6399
		3	----	15.2169
	C-C	1	----	8.2756
		2	----	12.4545
		3	----	17.4647
1.0	S-S	1	2.9919	2.9936
		2	6.2019	6.2051
		3	9.3674	9.3720
	C-F	1	----	1.1236
		2	----	7.1276
		3	----	25.7857
	C-S	1	----	4.3182
		2	----	7.6211
		3	----	10.8302
	C-C	1	----	6.0566
		2	----	8.895400
		3	----	12.45250

Table-7 : Non-Dimensional Critical Buckling Loads ($\hat{P}_{cr}$) of the Beam with Non-Nonlocal Theory of Elasticity

Nonlocal Parameter	Boundary Condition	Non-DimensionalCritical Buckling Load ($\hat{P}_{cr}$) (Wang et al. [28])	Non-DimensionalCritical Buckling Load ($\hat{P}_{cr}$) (Present)
0.5	S-S	1.6627	1.6630
	C-F	1.1041	1.1052
	C-S	--------	1.8197
	C-C	1.9035	1.9036
1.0	S-S	0.9079	0.9080
	C-F	0.7114	0.7115
	C-S	--------	0.9528
	C-C	0.9753	0.0753

and Wang et al. [28]. Present results of bending, vibration and buckling for all classical boundary conditions are listed in Tables-5 to 7 respectively. In these tables present results are compared with those available in the literature. From these tables one could observe that the present results are in good agreement with corresponding results reported in the literature. Present deflection results for CS and CC boundary conditions, frequency results for CS, CC and CF boundary conditions and critical loads for CS boundary conditions are new. These results for the above specific boundary conditions are not available in literature. From Table-5 it is observed that the maximum deflections for CC, CS, SS and CF boundary conditions are in increasing order. From Table-6 it is observed that the natural frequencies for CF, SS, CS and CC boundary conditions are in increasing order. From Table-7 it is observed that the critical loads for CF, SS, CS and CC boundary conditions are in increasing order. Further from Table- 5 to 7, it is observed that increase in nonlocal parameter leads to increase in deflection and decrease in natural frequency and critical buckling load. This is attributed to the fact that increase in nonlocal parameter decreases the effective stiffness of the nanotube.

Nonhomogeneous Nanotubes

Young’s modulus, nonlocal parameter, density and diameter of the nanotubes are assumed to vary along the axial direction. These parameters are expressed as

$$E = E_0\left\{1+k\left(\frac{x}{L}\right)^{\alpha}\right\},\ \mu = \mu_0\left\{1+k\left(\frac{x}{L}\right)^{\alpha}\right\},$$

$$\rho = \rho_0\left\{1+k\left(\frac{x}{L}\right)^{\alpha}\right\},\ d = d_0\left\{1+k\left(\frac{x}{L}\right)^{\alpha}\right\} \qquad (30)$$

E_0, μ_0, ρ_0 and d_0 are assumed to be the values of elastic modulus, nonlocal parameter, density and diameter at left end (x =0). E_0, μ_0, ρ_0 and d_0 are considered to be 1 TPa, 0.0136 nm^2 and 2.3 gm/cm^3 and 0.7 nm, respectively. Length (L) and wall thickness (t) of the nanotubes are considered to be 35 nm and 0.35 nm, respectively. In equation (38) α equals to 1, 2 and 3 represent linear, quadratic and cubic variations of the parameters, respectively. Eighty per cent variation of diameter (d) , Young’s modulus (E), nonlocal parameter (μ) and density (ρ) has been assumed in the nonhomogeneous analysis. SS boundary condition is considered for the nanotubes. Non-dimensional maximum deflections, natural frequencies and critical buckling loads are defined as follows

$$\overline{w} = w \times \frac{E_0 I_0}{qL^4},\ \ \overline{f} = f \times L^2\sqrt{\frac{\rho_0 A_0}{E_0 I_0}}\ \text{ and }\ \overline{P}_{cr} \times \frac{L^2}{E_0 I_0} \qquad (31)$$

These non-dimensional parameters are computed employing DQ method for S-S boundary condition. Effect of linear, quadratic and cubic variation of individual parameters has been investigated. Inter relation of these parameters are also investigated. Effect of linear, quadratic and cubic variation of elastic modulus for bending, vibration and buckling are shown in Figs.2-4, respectively. From these figures it is observed that maximum deflection decreases with increase in nonhomogeneous parameter. While natural frequency and critical load increase with increase in nonhomogeneous parameter. It has been shown that variations of w, f and P_{cr} are most severe for linear variation of E, μ, ρ, d and least severe for cubic variation and midway for quadratic variation. To explain this, let's consider the variation of maximum deflection (w) with Young's modulus (E). From elementary strength of materials results it is known that for homogeneous beam, maximum deflection decreases if Young's modulus increases. In the present work, we have considered elastic modulus of the non-homogeneous beam to be of the form:

$$E_x = E_0 \left\{ 1 + k \left(\frac{x}{L} \right)^{\alpha} \right\}$$

Let 's define the average Young's modulus as :

$$E_{avg} = \frac{\int_0^L E_x \, dx}{\int_0^L dx} = E_0 \left(1 + \frac{k}{\alpha + 1} \right) \tag{32}$$

It can be easily derived from Eqn. (2) that (a) for positive values of k, E_{avg} is largest for linear variation (α = 1) and lowest for cubic variation (α = 3) (b) for negative values of k (please note that $-0.8\, k < k < 0.8$), E_{avg} is smallest for linear variation and largest for cubic variation. So for positive values of k, deflection will be minimum for linear variation and maximum for cubic variation. Similarly, for negative values of k, deflection will be maximum for linear variation and minimum for cubic variation. This is reflected in Fig.2 , where variation of deflection can be seen to be most severe for linear variation of elastic modulus and least severe for cubic variation of elastic modulus.

Effect of linear, quadratic and cubic variation of nonlocal parameter for bending, vibration and buckling are shown in Figs.5-7, respectively. From Figs.5-7, one could observe that variation of nonlocal parameter has little effect on bending, vibration and buckling of nonhomogeneous nanotubes. So it can be concluded that though nonlocal parameter is considered to be an important factor for analysis of nano-structures, for nonhomogeneous nanotubes an average constant value of nonlocal parame-

Fig.2 Variations of Non-dimensional Maximum Deflections ($\overline{w}$) with Non-homogeneous Parameter (k) for Linear, Quadratic and Cubic Variations of Elastic Modulus

Fig.4 Variation of Non-dimensional Critical Buckling Loads ($\hat{P}_{cr}$) with Non-homogeneous Parameter (k) for Linear, Quadratic and Cubic Variations of Elastic Modulus

Fig.3 Variation of Non-dimensional Fundamental Natural Frequencies ($\overline{f}$) with Non-homogeneous Parameter (k) for Linear, Quadratic and Cubic Variations of Elastic Modulus

Fig.5 Variations of Non-dimensional Maximum Deflections ($\overline{w}$) with Non-homogeneous Parameter (k) for Linear, Quadratic and Cubic Variations of Nonlocal Parameter

Fig.6 Variation of Non-dimensional Fundamental Natural Frequencies $(\bar{f})$ *with Non-homogeneous Parameter (k) for Linear, Quadratic and Cubic Variations of Nonlocal Parameter*

Fig.7 Variations of Non-dimensional Maximum Critical Buckling Loads $(\hat{P}_{cr})$ *with Non-homogeneous Parameter (k) for Linear, Quadratic and Cubic Variations of Nonlocal Parameter*

Fig.8 Variations of Non-dimensional Fundamental Natural Frequencies $(\bar{f})$ *with Non-homogeneous Parameter (k) for Linear, Quadratic and Cubic Variations of Density*

ter can be considered for various nanotube applications and designs. This could reduce substantially the complexity of the formulation and computational effort. In Fig.8 effect of density on the vibration response of nanotubes has been shown. In this figure natural frequency shows greater rate of change for linear variation of nonhomogeneous parameter as compared to quadratic and cubic variations.

Effects of diameter of the nanotube on bending, vibration and buckling are shown in Figs.9-11, respectively. From these figures it is observed that maximum deflection decreases with increase in nonhomogeneous parameter. While natural frequency and critical load increase with increase in nonhomogeneous parameter. Further maximum deflection, natural frequency and critical load show greater rate of change for linear variation of nonhomo-

Fig.9 Variations of Non-dimensional Maximum Deflections $(\bar{w})$ *with Non-homogeneous Parameter (k) for Linear, Quadratic and Cubic Variations of Diameter*

Fig.10 Variations of Non-dimensional Fundamental Natural Frequencies $(\bar{f})$ *with Non-homogeneous Parameter (k) for Linear, Quadratic and Cubic Variations of Diameter*

Fig.11 Variations of Non-dimensional Critical Buckling Loads ($\hat{P}_{cr}$) with Non-homogeneous Parameter (k) for Linear, Quadratic and Cubic Variations of Diameter

Fig.13 Variations of Non-dimensional Fundamental Natural Frequencies ($\bar{f}$) with Non-homogeneous Parameter (k) for Linear Variation of Elastic Modulus (E), Nonlocal Parameter (μ), Density (ρ) and Diameter (d)

Fig.12 Variations of Non-dimensional Maximum Deflections ($\bar{w}$) with Non-homogeneous Parameter (k) for Linear Variation of Elastic Modulus (E), Nonlocal Parameter (μ) and Diameter (d)

Fig.14 Variations of Non-dimensional Critical Buckling Loads ($\hat{P}_{cr}$) with Non-homogeneous Parameter (k) for Linear Variation of Elastic Modulus (E), Nonlocal Parameter (μ) and Diameter (d)

geneous parameter as compared to quadratic and cubic variations. Effects of linear variation of elastic modulus, nonlocal parameter and diameter on bending, vibration and buckling has been shown in Figs.12-14, respectively. From Fig.12 it is observed that maximum deflection decreases with increase in nonhomogeneous parameter for linear variation of elastic modulus and nanotube diameter. Further it can be found that linear variation of diameter has stronger influence on the maximum deflection than the linear variation of elastic modulus.

From Fig.13 it is observed that natural frequency increases with increase in nonhomogeneous parameter for linear variation of elastic modulus and nanotube diameter. While natural frequency decreases with increase in nonhomogeneous parameter for linear variation of density of nanotube. Further it can be found that linear variation of diameter has stronger influence on the natural frequency than the linear variation of elastic modulus. From Fig.14 it is observed that critical buckling load increases with increase in nonhomogeneous parameter for linear variation of elastic modulus and nanotube diameter. Further it can be found that linear variation of diameter has stronger influence on the critical buckling load than the linear variation of elastic modulus. Maximum deflection, natural frequency and critical buckling load are observed to be most sensitive to change in nanotube diameter. While these are observed to be insensitive to the change in nonlocal parameter.

Conclusions

In this work, formulation and solutions methods are developed for nonhomogeneous single walled and double walled nanotubes. Analysis of nonhomogeneous nanotubes has been carried out employing differential quadrature method and nonlocal elasticity theory. Nonlocal theory has been implemented to consider the scale effect. Present results are validated with the results available in the literature for homogeneous nanotubes. Effect of linear, quadratic and cubic variations of nanotube Youngs modulus, nonlocal parameter, density and diameter on the structural response of the nonhomogeneous nanotubes is studied. It is observed that maximum deflection decreases with increase in nonhomogeneous parameter. While critical load increases with increase in nonhomogeneous parameter. Further maximum deflection, natural frequency and critical load show greater rate of change for linear variation of nonhomogeneous parameter as compared to quadratic and cubic variations.

It has been observed that the nonlocal parameter has little effect on the structural response of the nanotubes. While diameter, elastic modulus and density of the nanotubes have substantial effect on the response of the nanotubes. Extension of the present research work to incorporate shear deformation theories is under development.

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Appendix

$$\Omega_0^{bn} = -q(x)\left\{1-\frac{d^2\mu(x)}{dx^2}\right\} + 2\,\frac{d\mu(x)}{dx}\,\frac{dq(x)}{dx} + \mu(x)\,\frac{d^2q(x)}{dx^2}$$

$$\Omega_3^{bn} = \frac{d^2(E(x)\,I(x))}{dx^2}$$

$$\Omega_4^{bn} = 2\,\frac{d(E(x)\,I(x))}{dx}$$

$$\Omega_5^{bn} = E(x)\,I(x)$$

$$\Omega_3^{bk} = P\,\frac{d^2\mu(x)}{dx^2} - P + \frac{d^2(E(x)\,I(x))}{dx^2}$$

$$\Omega_4^{bk} = 2\left\{P\,\frac{d\mu(x)}{dx} + \frac{d(E(x)\,I(x))}{dx}\right\}$$

$$\Omega_5^{bk} = E(x)\,I(x) + P\,\mu(x)$$

$$\Omega_1^{vb} = -\left[\rho(x)A(x)\left\{1-\frac{d^2\mu(x)}{dx^2}\right\} - 2\frac{d(\rho(x)A(x))}{dx}\frac{d\mu(x)}{dx} - \mu(x)\frac{d^2(\rho(x)A(x))}{dx^2}\right]\omega^2$$

$$\Theta_3^{vb} = -\rho(x)A(x)\left\{1-\frac{d^2\mu(x)}{dx^2}\right\} + 2\frac{d(\rho(x)A(x))}{dx}\frac{d\mu(x)}{dx} + \mu(x)\frac{d^2(\rho(x)A(x))}{dx^2}$$

$$\Omega_2^{vb} = 2\left\{\mu(x)\frac{d(\rho(x)A(x))}{dx} + \rho(x)A(x)\frac{d\mu(x)}{dx}\right\}\omega^2$$

$$\Theta_5^{vb} = \mu(x)\rho(x)A(x)$$

$$\Omega_3^{vb} = \frac{d^2(E(x)I(x))}{dx^2} + \omega^2\mu(x)\rho(x)A(x)$$

$$\gamma_3^{vb} = \frac{d^2(E(x)I(x))}{dx^2}$$

$$\Omega_4^{vb} = 2\frac{d(E(x)I(x))}{dx}$$

$$\gamma_4^{vb} = 2\frac{d(E(x)I(x))}{dx}$$

$$\Omega_5^{vb} = E(x)I(x)$$

$$\gamma_5^{vb} = E(x)I(x)$$

$$\Theta_4^{vb} = 2\left\{\mu(x)\frac{d\rho(x)A(x))}{dx} + \rho(x)A(x)\frac{d\mu(x)}{dx}\right\}$$