PARAMETER OPTIMIZATION WITH CHEBYSHEV POLYNOMIALS FOR TRAJECTORY DESIGN OF A RLV

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Abstract

A trajectory optimal control problem is converted to a parameter optimization problem by approximating the states and control using piecewise Chebyshev polynomials. The Chebyshev points which control the polynomial fit is used to match the dynamics at the nodal points. Midpoint strategy significantly reduced the number of parameters to be optimized without having to sacrifice on accuracy. The resultant non- linear programming problem is solved using sequential quadratic programming method. Ascent phase trajectory optimization of a reusable launch vehicle having typical path and terminal constraints is taken as a test case.

Keywords: *collocation, Chebyshev polynomial, sequential quadratic programming, reusable launch vehicle*

Introduction

The solution of optimal control problem using Euler-Lagrange equation or co- state variables methods is cumbersome due to sensitivity of boundary values which results jump in co-state variable and other associated numerical difficulties. An alternate strategy could be to convert the optimal control problem into a parameter optimization problem. Collocation and differential inclusion are examples of such conversion methods [Ref.1]. In collocation method, states and control are approximated by piecewise continuous polynomials. The differential equations are replaced by defect constraints, which are driven to zero. The program NPDOT [Ref.2] converts the optimal control problem to a nonlinear programming problem using *Hermite-Simpson* defect constraints. Using Hermite interpolation, cubic polynomials are defined for each of the states that are matched along with its derivative, as defined by the system dynamics, at the nodal points. The values of the states at the nodes are the design variables for the NLP problem, which force the defect (of the interpolated derivative and differential equations) to zero. In another technique, referred to as differential inclusion [Ref.3], control variables are removed by defining a bound on the rate of change of states. Advantages of such a method are reduced problem size, which helps in saving computational time. Some of the disadvantages of differential inclusion method [Ref.4] are increased difficulty in the analytical derivation of the constraints with respect to problem parameters, restriction of problems with linearly varying controls and the requirement of obtaining control time histories. In [Ref.5], pseudo spectral method is proposed which gives better approximation of discretization of the derivative of states at the nodes, thereby making it competitive with the direct collocation methods.

In the present paper, states and control are approximated by piecewise Chebyshev polynomials. Derivative of these polynomials are matched with the system dynamics at the nodal points. The resultant parameter optimization problem is then solved using Sequential Quadratic Programming (SQP) approach. The idea is similar to the method described by [Ref.6] in which differentiation and integration are performed in closed form and the variable thrust trajectory problem is reduced to one of the ordinary calculus. Newton's method is then used to solve this parameter optimization problem. In [Ref.7] a penalty function approach is used to satisfy the system dynamics at nodes. As the name suggests, a penalty function penalizes the performance index for any constraint violation [Ref.8]. Another approach used Chebyshev polynomials to approximate the control and used second order method with a penalty function to solve the parameter optimization problem [Ref.9]. In [Ref.2] Hermite-interpolation implicit integration scheme is used to convert optimal control problem into a parameter optimization one. Higher order Gauss-Lobatto methods are reported in [Ref.10].

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A trajectory optimal control problem converted to a parameter optimization problem with Chebyshev polynomials will lead to a large number of equality constraints. In addition there will be problem dependent path constraints. Such a problem is computationally expensive to solve. Initial attempt was made to reduce the problem size, so as to improve upon the implicit integration step size, lead to non-convergence, which meant system dynamics not getting satisfied at nodal points. An alternate, mid point strategy is proposed which required additional constraints to be satisfied at the center of the segment. Though initially it looked that problem size is merely increased, the additional computation improved the accuracy of polynomial for a larger segment. The implicit integration step size is enhanced and the overall size of the problem is brought down to one-sixth of its earlier size. This methodology is applied to trajectory optimization problem of a Reusable Launch Vehicle (RLV).

A typical RLV demonstrator flight profile could be boosting the RLV to hypersonic Mach number using some propulsion system. The RLV , after getting separated from the booster, will perform sub-orbital flight and subsequently recovered from sea using a parachute and floatation systems. Ascent phase trajectory optimization of a RLV along with the booster is taken as a test case. Typical constraints during ascent phase will be dynamic pressure (q) and $q\alpha$ (α is angle of attack and is used as control variable). The term $q\alpha$ has direct bearing on the loads coming on the vehicle. Terminal constraint is on burnout flight path angle (γ). This parameter plays an important role during the descent phase. A large γ at burnout implies steeper reentry, which will violate the descent phase constraints such as load factor and q. Performance index for this problem is to maximize the Mach number at the burn out of the booster. The optimal control problem is converted to a parameter optimization problem using Chebyshev polynomials and the proposed mid point strategy successfully applied.

Chebyshev Polynomials

Polynomial interpolation using equally spaced data results in accurate computation in the middle range of the interpolation domain but the error of interpolation increases towards the corners. In Chebyshev polynomials the spacing is largest at the center of interpolation domain and decreases towards comers. So error is more evenly distributed throughout the domain and magnitude of error is less than that of equally spaced parts [Ref.11 to 15].

Starting with $T_0(x) = 1$ and $T_1(x) = x$, Chebyshev polynomial of any higher order in power series can be generated by the recurrence relation

$$T_N(x) = 2x\, T_{N-1}(x) - T_{N-2}(x) \tag{1}$$

In equivalent cosine function form, these polynomials can be expressed as

$$T_N(x) = \cos\,(N \cos^{-1}(x)) \qquad -1 \le x \le 1 \tag{2}$$

$T_N\,(x)$ have the value unit at $x = 1$, and at $x = -1$the value is +1 for even N and -1 for odd N. The k^{th} root of the Chebyshev polynomial is given by

$$x_k = \cos\left(\frac{2k + 1\pi}{N + 1\ 2}\right) \tag{3}$$

Chebyshev polynomials are orthogonal polynomials. Polynomials $\phi_i(x)$ are said to be mutually orthogonal with respect to weight function $w(x)$ when

$$\int_{-1}^{1} w(x)\, \phi_i\,(x)\, \phi_j\,(x)\, dx = 0 \qquad , i \ne j \tag{4}$$

where $w(x) = (1–x^2)^{-\,½}$ for such polynomials. The roots are real and distinct for orthogonal polynomials. Also as they are linearly independent, any such polynomial is a linear combination of the basis polynomials.

As the states and control is to be approximated by polynomials, our aim is to find a polynomial of degree N which fits $f\,(x)$ exactly at data points x_k with $k = 0,1,. . . , N$. Such a polynomial is given by

$$P_N\,(x) = \sum_{r=0}^{N} C_r\, T_r\,(x) \tag{5}$$

where

$$C_r \;=\; \frac{2}{N+1} \sum_{r=0}^{N} f\,(x_k)\; T_r\,(x_k) \tag{6}$$

The data points x_k are zeros of the Chebyshev polynomial $T_{N+1}(x)$ and over the complete range $-1 \le x \le 1$. The error $e_N\,(x) = f\,(x) - P_N\,(x)$ satisfies for sufficiently smooth functions the *minimax* criterion

$$\max \left| e_N(x) / f^{N+1}(\xi) \right| = \min. \tag{7}$$

Where ξ is some point in (-1, 1).

Mathematical Formulation

The trajectory equation of motion is governed by first order differential equations

$$\dot{x} = f(x, u, t) \tag{8}$$

where x is a vector of state and u is a vector of control.

One method of solution could be to discretize the time t in i equal or unequal parts, approximate the state and control with polynomials such that dynamics is satisfied at the nodal points (t_i, t_{i+1}). This solution method is called as *collocation*.

Let the state be approximated by a cubic polynomial

$$x = c_0 + c_1\tau + c_2\tau^2 + c_3\tau^3 \quad , \tau \in [t_i, t_{i+1}] \tag{9}$$

Let x_1, x_2, x_3 and x_4 be the states computed at the Chebyshev points generated between (t_i, t_{i+1}). Coefficients of this polynomial are determined using equation (6). The first derivative of equation (9) is then matched with the dynamics at the nodes.

$$c_1 + 2c_2\, t_i + 3c_3\, t_i^2 = f(x_i, u_i, t_i) \tag{10}$$

$$c_1 + 2c_2\, t_{i+1} + 3c_3\, t_{i+1}^2 = f(x_{i+1}, u_{i+1}, t_{i+1}) \tag{11}$$

where x_i and x_{i+1} are the states computed at the nodes using equation (9). In addition states are to be matched at the nodes along with their slopes for two successive polynomials to ensure that these are continuous. Determining x_1, x_2, x_3 and x_4,which will satisfy the equality constraint in equations (10) and (11) will give a good approximation of state satisfying the dynamics at (8). Addition of another constraint equation (12)

$$c_1 + 2c_2\, t_m + 3c_3\, t_m^2 = f(x_m, u_m, t_m) \tag{12}$$

improves the results significantly, as a larger step size could be taken up during the implicit integration process.

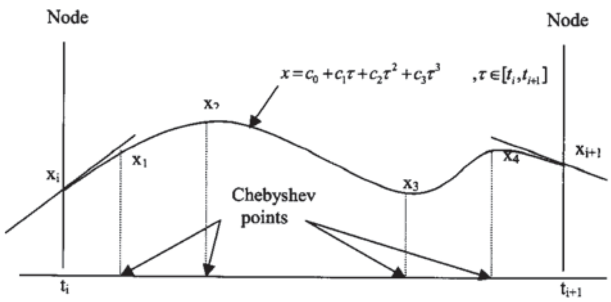

Fig. 1 Implicit integration scheme

Here $t_m = \dfrac{(t_i + t_{i+1})}{2}$ and x_m is the vector of state computed at the center of node using (9).

It is assumed that a cubic polynomial will be able to approximate vector of states and control (Fig.1). This requires four parameters (k = 4) to be estimated for each state at the roots of the Chebyshev polynomial in the time span t_i to t_{i+1}. Let there be N such polynomials, which is required to be fitted for a given state. There will be then $(N + 1)$ nodes for this state. States and slopes are matched at each of the nodes with the subsequent polynomial and also simultaneously satisfying the dynamics. The polynomial should also satisfy dynamics at the mid point. Thus there will be 5 equality constraints to be satisfied per polynomial per state. For the control, slope condition is relaxed and only numerical value is matched at the nodes. Let S be the number of differential equations to be satisfied. Thus number of parameters to be determined is $(S\,N\,k)$ and number of equality constraints to be satisfied is $(5N\text{-}1)S$. If the equality constraints can be completely satisfied, then the piecewise polynomials will give a good approximation of the dynamics. The path constraints, which are problem dependent, are forced at the nodes and Chebyshev points. The resultant parameter optimization problem is then solved using Sequential Quadratic Programming (SQP) method [Ref.16-19]. In SQP method, performance index is approximated by a quadratic function and constraints are linearized.

Trajectory Optimization

The equations of motion as derived in report [Ref. 20], for a point mass, non- rotating earth model is given by

$$\dot{V} = -g \sin\gamma + \frac{(T\cos\alpha - D)}{m}$$

$$\dot{\gamma} = \left(\frac{V}{r} - \frac{g}{V}\right)\cos\gamma + \frac{(L + T\sin\alpha)}{mV}\cos\sigma$$

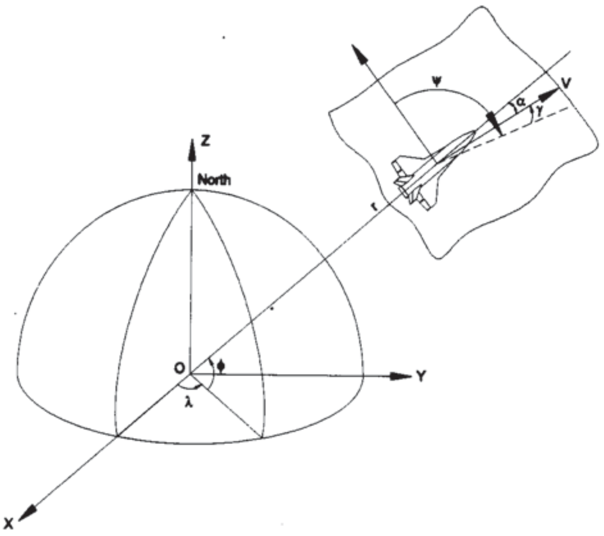

Fig. 2 Definition of paratmeters

$$\dot{\psi} = -\frac{V}{r}\cos\gamma \sin\psi \tan\phi + \frac{(L+T\sin\alpha)}{mV}\frac{\sin\alpha}{\cos\gamma}$$

$$\dot{r} = V\sin\gamma$$

$$\dot{\lambda} = \frac{V}{r}\frac{\cos\gamma \sin\psi}{\cos\phi}$$

$$\dot{\phi} = \frac{V}{r}\cos\gamma \cos\psi \qquad (13\text{-}18)$$

where $V, \gamma, \psi, r, \lambda$ and ϕ are the state variables corresponding to velocity, flight path angle measured from local horizontal, heading angle measured from true north, radial vector, longitude and latitude respectively (Fig.2). Angle of attack α and bank angle σ are the control variables. As only planar trajectory is considered in the present analysis $\sigma = 0$ is assumed. Thrust, lift and drag forces are given by T, L and D. Indian Standard Atmosphere is used where atmospheric properties such as density, pressure and temperature are stored as a function of altitude. Density (ρ) will be used for computation of dynamic pressure ($q = ½\, \rho V^2$),which will be required, by lift ($L = qSC_{N\alpha}\, \alpha$) and drag ($D = qSC_D$) equations. C_D is the drag coefficient and C_{Na} is normal force coefficient slope. S is a reference area for which these coefficients are derived. Pressure will be used for thrust correction and temperature for the computation of speed of sound. Gravity is computed using the expression $g = g_0\, (\frac{r_0}{r})^2$ where g_0 (9.80665 m/s^2) is the acceleration due to gravity at the earth surface and r_0 (6378 km) is the mean radius of earth. Initial state vectors [$V, \gamma, \psi, r, \lambda, \phi$] are taken as [1 m/s, 90º, 90º, 6m, 80.2º, 13.7º]. Initial altitude of 6m corresponds to center of gravity of the vehicle. Initial velocity is not taken as zero to avoid numerical difficulties in equations (14) and (15).

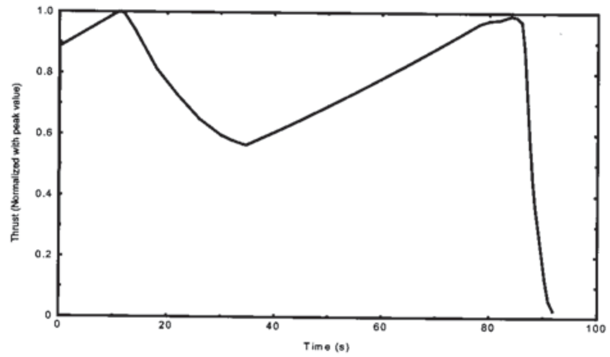

Fig. 3 Thrust-time history of the Booster

The constraint trajectory optimization problem during the thrusting ascent phase can be stated as

Maximize

$$M_{tf} \qquad (19)$$

subject to

$$q \le 50 \quad (kPa) \qquad (20)$$

$$|q\alpha| \le 250 \quad (Pa\ rad)$$

$$\gamma_{tf} = \quad (deg) \qquad (21)$$

with bounds

$$-2 \le \alpha \le 2 \quad (deg) \qquad (22)$$

where M_{tf} is the Mach number at burnout of the booster ($at\ t = t_f$). The dynamic pressure constraint ensures that power requirements by the booster fin actuators are within limits. Similarly $q\alpha$ constraint limits the structural load on the vehicle. Nominal trajectory is defined with no winds. With winds $q\alpha$ will be much higher. The burnout condition of flight path angle ensures that reentry constraints are within limits. Again, the control variable bounds are for no wind conditions.

Results

The thrust-time curve of the booster, which is normalized to its peak thrust, is shown in Fig.3. Normal force coefficient and drag as a function of Mach number is shown in Fig.4. This is for a reference area of 6m^2. These are inputs to the propulsion and aerodynamic model.

Table-1 : Results with Different Number of Polynomials Approximating the State

N	Number of Parameters	Number of Equality Constraints	Path Constraints + Terminal Constraints	Performance Index	$\|\|c\|\|_2$
8	224	200	81	-	>>1
9	252	225	91	6.446	2.9 x 10^{-7}
10	280	250	101	6.482	1.1 x 10^{-7}
11	308	275	111	6.483	1.0 x 10^{-7}

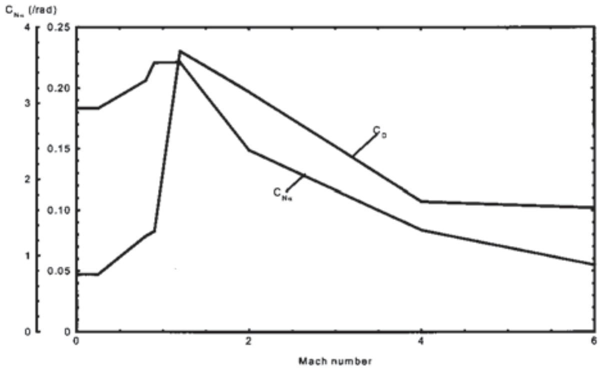

Fig. 4 Aerodynamic parameters during Ascent Flight

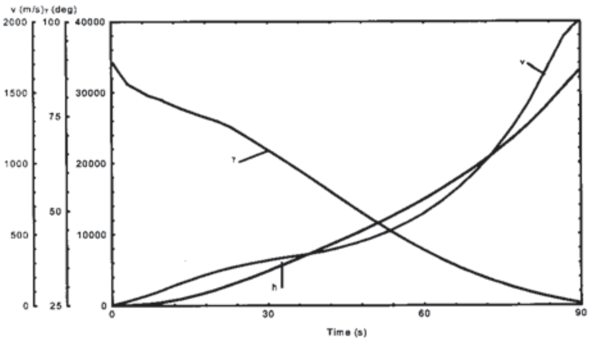

Fig. 5 Important state variables

In order to reduce the number of equality constraints, which are formulation dependent, initial attempts were made to solve the parameter optimization problem without satisfying equation (12). The implicit integration step size achieved was of the order of 2*s*. For the trajectory optimization problem of 90*s* duration, as in the present case, the required N that will completely satisfy the dynamics is 45. Thus for S=7 (six state equations as in 13-18 and one control equation), N=45 and k=4, the optimization problem has to determine 1260 parameters and satisfy 836 ((3S-2) N) equality constraints. In addition there will be path constraints. Such a problem is computationally expensive to solve. Any scheme, which drastically reduces N without having to sacrifice on accuracy, will be a pragmatic way to solve such problems. The mid point strategy, in which additional constraints are to be satisfied at the center, significantly reduced the value of N to 9. This implies that implicit integration step size is enhanced to $10s\left(\frac{90}{N}\right)$. The overall size of the parameter optimization is brought down to 252 parameters and 225 ((4S - 3) N) equality constraints. This is roughly one-sixth the size of the earlier problem and therefore much easier to solve. The physics of mid point approach is that with additional computation one can now enlarge the segment length of the approximating polynomial for the same accuracy. Though initially it looked that problem size is getting increased, the additional computation improved the accuracy of polynomial for a larger segment length. The implicit integration step size is enhanced and the overall size of the problem is brought down. Below N = 9, the L_2 norm of equality constraint vector had a large value thereby indicating that dynamics is not satisfied by the piecewise polynomials. The L_2 norm of a vector is defined as the root sum square of its components. Above N = 9, performance index and L_2 norm of equality constraint vector ($||c||_2$) did improve with increasing N. The improvement is however marginal as given in Table-1.

The history of three important state parameters viz. altitude, velocity and flight path angle are shown in Fig.5. Terminal constraint on γ is satisfied. Fig.6 shows the constraints q and $q\alpha$. Both the constraints are active during certain phase of time. As the slope matching condition is relaxed for the control, sharp comers are observed at the nodal points in the control history (Fig.7).

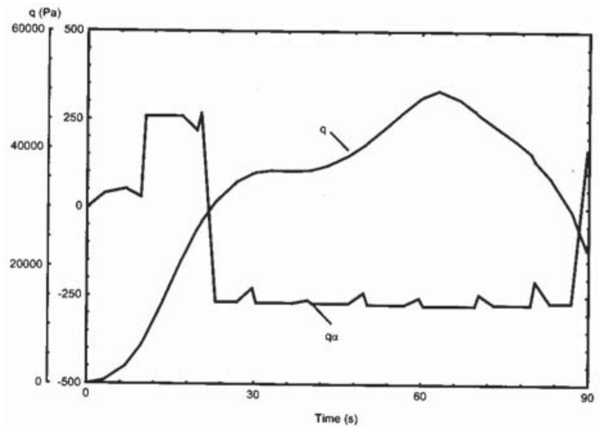

Fig. 6 Constraint history

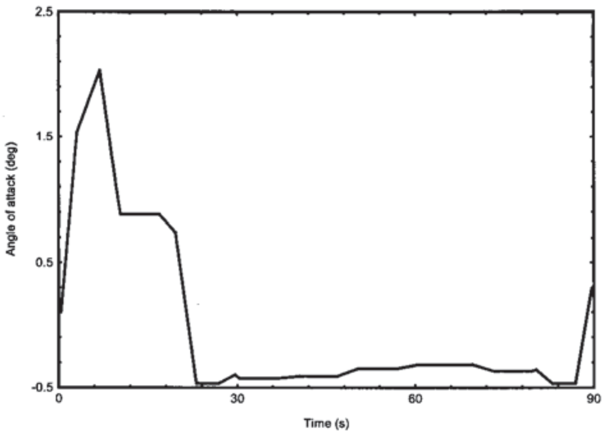

Fig. 7 Control history

Conclusions

A trajectory optimal control problem is converted to a parameter optimization problem by approximating the states and control using piecewise Chebyshev polynomials. The case study is on the ascent phase trajectory optimization of a RLV flight, which has typical path constraints such as dynamic pressure and $q\alpha$. The constraints are satisfied by the optimization strategy. The performance index improved with more number of polynomials at the cost of increasing computational time. The Chebyshev points which control the polynomial fit is used to match the dynamics at nodes. Midpoint strategy significantly reduced the number of parameters to be optimized without having to sacrifice accuracy.

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