NOVEL METHOD FOR COARSE ALIGNMENT OF STRAPDOWN INS ON OSCILLATORY BASE

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Abstract

Novel Analytic Coarse Alignment Method is presented for Strapdown Inertial Navigation Systems (SDINS) alignment on oscillatory base. Oscillatory base alignment is conducted by building two vectors derived from gravity and angular velocity information provided by the strapdown INS under oscillatory motion and comparing them with similar vectors built using the local level reference frame information. The theory and algorithms are detailed in the paper and several simulation test results are compiled for various kinds of oscillatory motion.The proposed algorithm could be used for sea based INS applications.

Nomenclature

C_j^i = direction cosine matrix from frame *j* to frame *i*
$\hat{C}_j^i$ = approximate direction cosine matrix from frame *j* to frame *i*
x = vectorial quantity $\{x_1, x_2, x_3\}^T$
$\hat{x}$ = noisy vector of *x*
ϕ, θ, ψ = Euler angles of roll, pitch and yaw/ azimuth/heading

Superscripts

n = navigation frame/local-level frame
ni = an inertia frame co-aligned to navigation frame at some instant *k*
b = body frame hosting orthogonal triad of inertial sensors
bi = an inertial frame co-aligned to body frame at some instant *k*

Introduction

Today the SDINS, with or without external aids, are employed in several marine related applications. The alignment process is an essential precursor for the start of navigation with any SDINS irrespective of the environment. The alignment accuracy has immense bearing on the navigation performance eventually. Initial attitude misalignment errors compound with time resulting in exponential error growth in position [1]. The critical factors that influence the accuracy of the coarse alignment procedure are the alignment duration and the accuracy of the inertial sensors.

This paper presents a novel method for alignment of Strapdown Inertial Navigation Systems (SDINS) on oscillatory base. The method is useful for a variety of marine SDINS applications.

Traditionally, the alignment is conducted in two parts comprising of coarse alignment and fine alignment. The purpose of coarse alignment is to provide as close possible an estimate of the initial attitude for the initialization of the fine alignment procedure. And the fine alignment procedure [4] estimates the gyro biases and corrects the coarse attitude towards the true value subject to the measurement noise of the inertial sensors. Design of fine alignment scheme normally assumes the attitude misalignment to be small, in the order of 1°. They, hence, rely heavily on the initial coarse attitude estimate. Without such an estimate, the fine alignment filter would fail to converge. In other globally convergent attitude optimization schemes [5], the coarse estimate could be used to minimize the convergence time of the global attitude estimator.

An analytic coarse alignment scheme for the SDINS on oscillatory base is presented with this paper. The necessary background is built systematically and derivations borrowed from previous literature are re-done for the sake of completeness. The emphasis of this paper would be to obtain mathematically closed forms of solution to the oscillatory base alignment problem.

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The scheme is put to test with an high grade INS system mounted on two-axis rate table. The statistical results of the attitude accuracy, particularly that of the azimuth, are presented and discussed.

Principle of Coarse Alignment

The generic requirement for self-alignment of SDINS is to find two non-collinear vectors simultaneously in the two frames of reference between whom the attitude transformation is desired to be obtained. Two non-collinear vectors are sought, and together with their cross product are sufficient to achieve the desired 3-D transformation. Normally in the case of SDINS alignment for marine applications, one of the two frames is the body frame and the other is the navigation or local-level frame. The analytic alignment problem becomes to build a set of three linearly independent vectors $\{V_1, V_2, V_3\}$ with $V_3 = V_1 \times V_2$ upon body and navigation frames such that the solution of the transformation matrix C_b^n to

$$\left[V_1^n, V_2^n, V_3^n\right] = C_b^n \left[V_1^b, V_2^b, V_3^b\right] \tag{1}$$

exists. Using simple matrix algebra, the transformation matrix C_b^n can be directly obtained from Eqn.1 as

$$C_b^n = \left[V_1^n, V_2^n, V_3^n\right] \left[V_1^b, V_2^b, V_3^b\right]^{-1} \tag{2}$$

Alternatively, since C_b^n is orthogonal,

$$(C_b^n)^{-1} = (C_b^n)^T \tag{3}$$

and hence C_b^n can also be written as

$$C_b^n = \begin{bmatrix} (V_1^n)^T \\ (V_2^n)^T \\ (V_3^n)^T \end{bmatrix}^{-1} \begin{bmatrix} (V_1^b)^T \\ (V_2^b)^T \\ (V_3^b)^T \end{bmatrix} \tag{4}$$

However, since the body frame measurements are polluted with noise, the transformation matrix arrived from Eqn.(4) is only approximate. The coarse DCM $\hat{C}_b^n$ due to noisy $\left\{V_{k=1,2,3}^b\right\}$ is given as

$$\hat{C}_b^n = \begin{bmatrix} (V_1^n)^T \\ (V_2^n)^T \\ (V_3^n)^T \end{bmatrix}^{-1} \begin{bmatrix} (\hat{V}_1^b)^T \\ (\hat{V}_2^b)^T \\ (\hat{V}_3^b)^T \end{bmatrix} \tag{5}$$

Eqn. (5) is the general closed-form solution of the attitude determination problem given two non-collinear vector measurements. Eqn. (5) is also the often referred form in literature and considered computationally superior to the form in Eqn. (2) as the inverse $\left(\left[V_1^{n^T}, V_2^{n^T}, V_3^{3^T}\right]^T\right)^{-1}$ can be pre-computed owing to the prior knowledge of the navigation frame vectors. The vectors in the b-frame $\left\{V_{k=1,2,3}^b\right\}$ are formed from the measurements of the body mounted orthogonal triad of gyros and accelerometers. The noise plaguing the body sensor measurements can be minimized by simple averaging. The duration for averaging is such minimum time after which the first moment of the body measurements becomes stationary, normally about 2-3mins. The other two preconditions for satisfactory estimation of the direction cosine matrix (DCM) $\hat{C}_b^n$ are as follows :

- The matrix product $V_1^{n^T} V_2^n$ should be as close to 0 as realistically possible by a suitable choice of V_1^n and V_2^n. The condition of $V_1^{n^T} V_2^n \equiv 0$ gives the maximum observability to the attitude estimation.
- The alignment is not undertaken at the poles of the Earth as the North merges with the vertical axis of the navigation frame and a degree of freedom is lost. The vectors $\{V_1, V_2, V_3\}$ become linearly dependent. This is similar to the gimbal lock problem in gimballed INS systems.

Coarse Alignment on Oscillatory Base

The transformation matrix C_b^n by definition transforms a unique physical quantity between the *b* and *n* frames. As discussed earlier, for the purpose of coarse alignment, the physical quantity should also be amenable to measurement in both the frames simultaneously. Of the two measured quantities by the SDINS, the gravity vector g^b and angular rate vector ω^b, we note that the magnitude of gravity vector is unaltered under oscillation. However, its direction changes such that $\hat{g}^b = \hat{C}_n^b g^n$. The gravity vector in n frame, g^n, neither alters in magnitude nor changes direction, while the SDINS is stationed on the constant latitude circle. The angular rate vector is not comparable between the b and n frames, while SDINS is under oscillation as the body measured inertial angular rate $\hat{\omega}_{ib}^b$ is substantially higher than and deviated from the Earth angular rate, $\hat{\omega}_{in}^n$. We now develop a novel method of coarse alignment for the SDINS under oscillation by constructing suitable non-collinear vectors derived from the ***g*** vector in the body and navigation frames. Few preliminaries are revisited hereunder for the sake of completeness.

Time propagation of DCM Matrix

Let us assume that at instant k, the approximate DCM $\hat{C}_b^n(k) \equiv \hat{C}_{bk}^{nk}$ is known and that it indeed transforms

$$g^n(k) = \hat{C}_b^n(k)\,\hat{g}^b(k) + \xi(k) \tag{6}$$

then the coarse DCM at any further instant $l > k$ can be obtained using the gyro output vector ω_{ib}^b and Earth angular rate ω_{in}^n that indeed transforms

$$g^n(k+l) = \hat{C}_b^n(k+l)\, g^b(k+l) + \xi(k+l) \tag{7}$$

and the time propagation of C_b^n between instants k and $k + l$ is obtained as

$$\hat{C}_b^n(k+l) = C_{n_k}^{n_{k+l}}\, \hat{C}_b^n(k)\, \hat{C}_{b_{k+l}}^{b_k} \tag{8}$$

with $\xi(k)$ and $\xi(k+l)$ being the small difference between the transformed gravity vector and the true gravity vector arising mainly due to the approximate $\hat{C}_b^n$ at instants k and $k + l$ respectively.

The construction of time propagation matrices $C_{n_k}^{n_{k+l}}$ and $\hat{C}_{b_{k+l}}^{b_k}$ in the n and b frames respectively is given in Appendix.

Observers in Inertial Frame

Let an hypothetical inertial frame be co-aligned with the navigation frame n at the instant k, the start point of alignment process. Denote the inertial frame as ni. For an observer located in the inertial frame ni, the loci of gravity vectors arising due to the rotation of the Earth describes a cone and hence at no two instants the direction of gravity is same for the inertial observer in a full day duration.

The expression for gravity vectors as seen by the inertial observer over time is obtained by transforming the gravity vector at instant $k + l$ in the navigation frame n to the inertial frame ni through the transformation

$$g^{ni}(k+l) = C_{n_{k+l}}^{n_k}\, g^n(k+l) \tag{9}$$

Similarly, for an observer in an another hypothetical inertial frame bi co-aligned with the SDINS body from b at the start of alignment process, the loci of gravity vectors is obtained from

$$g^{bi}(k+l) = \hat{C}_{b_{k+l}}^{b_k}\, \hat{g}^b(k+l) \tag{10}$$

At the start of alignment process, since it is assumed that the inertial frames bi and ni are co-aligned with the body frame b and navigation frame n at the instant k, the following DCM relations are true.

$$\hat{C}_{bi}^{ni} \equiv \hat{C}_b^n(k) \equiv \hat{C}_{b_k}^{n_k} \tag{11}$$

We now prove that irrespective of SDINS oscillation, the profile of gravity vectors viewed by the inertial observers in the bi and ni frames is always related by the constant DCM matrix $\hat{C}_{bi}^{ni}$.

Statement : At any instant m , $m \geq 0$,

$$g^{ni}(k+m) = \hat{C}_{bi}^{ni}\, \hat{g}^{bi}(k+m) \tag{12}$$

Proof : By definition, $C_b^n(k+m) \equiv C_{b_{k+m}}^{n_{k+m}}$, relates vectors in b and n frames at instant $k + m$ as

$$g^n(k+m) = \hat{C}_b^n(k+m)\, \hat{g}^b(k+m)$$

Bringing the vectors $g^n(k+m)$ and $\hat{g}^b(k+m)$ to their respective inertial frames using Eqn. (9) and Eqn. (10).

$$C^{n_{k+m}}_{n_k}\, g^{ni}(k+m) = \hat{C}^{n}_{b}(k+m)\, \hat{C}^{b_{k+m}}_{b_k}\, \hat{g}^{bi}(k+m)$$

$$g^{ni}(k+m) = C^{n_k}_{n_{k+m}}\, \hat{C}^{n_{k+m}}_{b_{k+m}}\, \hat{C}^{b_{k+m}}_{b_k}\, \hat{g}^{bi}(k+m)$$

$$g^{ni}(k+m) = \hat{C}^{n_k}_{b_k}\, \hat{g}^{bi}(k+m)$$

Recalling from Eqn (11) that $\hat{C}^{n}_{b}(k) \equiv \hat{C}^{ni}_{b}$, we arrive at the equation

$$g^{ni}(k+m) = \hat{C}^{ni}_{bi}\, g^{bi}(k+m) \tag{13}$$

and that is the desired proof.

It is understood from the preceding arguments that the inertial observers in *bi* and *ni* frames are looking at the same loci of gravity vectors, albeit with the different attitude, even as the body frame is oscillating relative to the navigation frame. We use this singular fact and propose to solve the attitude determination problem to estimate the DCM $\hat{C}^{ni}_{bi}$ and proceed to propagate the estimate $\hat{C}^{ni}_{bi}$ to the present using attitude propagation described in Eqn. (8) to obtain $\hat{C}^{n}_{b}$ in the present.

Novel Coarse Alignment Scheme : Mid Frame Method

In this method, two resultant gravity vectors are constructed on either side of the inertial frames of reference *bi* and *ni*. The alignment duration is between the instants k and $k + n$, with the number of intervening samples n assumed to be even for sake of clarity of expression.

Let us associate the two inertial frames *bi* and *ni* to be co-aligned with the body and navigation frames at instant $k+\frac{n}{2}$, ie., at the mid-point of the arc describing the loci of gravity vectors during the alignment period, such that there are $\frac{n}{2}$ gravity vectors on either side of the inertial reference axes. For the observer in the *ni* frame, due to the rotation of the Earth, the gravity vectors seem to be coming from the West and going towards the East with one gravity vector coinciding with him at the instant $k+\frac{n}{2}$. Let us call the resultant of gravity vectors coming from the West of the inertial observer as g_{west} and the resultant of gravity vectors going to the East as g_{east}. The vectors g_{west} and g_{east} subtend an angle at the midpoint. Together with their cross product, $g_{west} \times g_{east}$, they form a basis for the 3-D space and hence are suitable candidates for coarse attitude estimation.

Formulae for constructing the vectors g_{west} and g_{east} in both navigation and body frames are as given below :

Navigation frame vectors :

$$V^{n}_{1} = g^{ni}_{west} = \sum_{l=0}^{\frac{n}{2}} C^{n_{(k+\frac{n}{2})}}_{n_{k+l}}\, g^{n}(k+l) \tag{14}$$

$$V^{n}_{2} = g^{ni}_{east} = \sum_{l=\frac{n}{2}}^{n} C^{n_{(k+\frac{n}{2})}}_{n_{k+l}}\, g^{n}(k+l) \tag{15}$$

Body frame vectors :

$$V^{b}_{1} = g^{bi}_{west} = \sum_{l=0}^{\frac{n}{2}} C^{b_{(k+\frac{n}{2})}}_{b_{k+l}}\, g^{b}(k+l) \tag{16}$$

$$V^{b}_{2} = g^{bi}_{east} = \sum_{l=\frac{n}{2}}^{n} C^{b_{(k+\frac{n}{2})}}_{b_{k+l}}\, g^{b}(k+l) \tag{17}$$

With the tedious part of forming the vectors $\{V_1, V_2, V_3\}$ complete, it only remains to substitute them in Eqn. (5) to obtain the DCM relating the inertial frames $\hat{C}^{ni}_{bi}$. The $\hat{C}^{ni}_{bi}$ is then propagated using Eqn. (8) to get the DCM $\hat{C}^{n}_{b}$ relating the body frame with navigation frame at the latest instant. Following observations are made on the proposed method.

- For an alignment duration of t secs., the vector dot product of g_{east} and g_{west} is $g_{east} \bullet g_{west} = \sin^2(L) + \cos^2(L)\cos(\|\omega_{ie}\|\frac{t}{2})$ between them. ω_{ie} is the Earth rate vector and L the latitude. This formula could indeed be used to estimate the latitude itself, at an unknown location given the sensor inputs.

The method is entirely recursive with minimal data storage requirements. The resultant vectors in navigation frame $\left\{g^{n}_{east}, g^{n}_{west}\right\}$ can be pre-computed with the prior knowledge of g_n and ω^{n}_{ie}.

The Fig. 1 pictorially depicts the scheme for the special case of SDINS located on the equator. The equitorial case is chosen to render the diagram simple and easy to understand.

Results

Several simulation studies were conducted to test the performance of the proposed algorithm. The results are presented hereunder along with the true values (Table-1). The test bed comprised of simulated three-axis angular motion with no noise or bias to the gyro output.

The method is also implemented and tested on several different INS platforms with encouraging results. The chosen SDINS for the purpose of present analysis consists of 0.01 - 0.05°/hr triad of gyros and 100 - 150 μg class of accelerometers mounted in orthogonal configuration. The SDINS is mounted on a two-axis rate table with induced coning motion of $2 \sin (2\pi\ 0.25\ t)$ on the roll and pitch axes of the INS. The results of error in Euler angles are produced in Table-2 for analysis purposes.

Discussion on the results

By the nature of the method, the sensor noises and numerical errors accumulate over time degrading the alignment accuracy. This would tempt to lead the designer to allow relatively little time for alignment. However, too short time for alignment would bring the two vectors uncomfortably close to each other resulting in almost singularity. Hence, a suitable trade off between alignment time vis-a-vis sensor noise levels have to be made for improving the accuracy of alignment specific for the application. This decision is mostly empirical and in the choice of the designer.

Conclusion

A novel coarse alignment method is presented for the strapdown inertial navigation system on oscillatory base. Accuracy results for the Euler angles for SDINS under oscillation are presented. The achieved accuracy in all three Euler angles ψ, ϕ and θ are found to be within acceptable levels for the initiation of fine alignment procedure.

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Table-1 : True and Simulated Values of ϕ ,θ and ψ angles

	ϕ yaw	θ roll	ψ pitch	input excitation (radians/s)
true	0	0	0	0 sin (t)
simulated	0	0	0	
true	0.0000000000	0.8094872042	0.0000000000	0.1 e-01 sin (t) in roll axes
simulated	-0.0000001145	0.8094872228	0.0000002537	
true	0.0000000000	0.0000000000	0.8094872042	0.1 e-01 sin (t) in pitch axes
simulated	0.0000007210	-0.0000002570	0.8094872203	
true	-0.0057186852	0.8094602221	0.8095410155	0.1 e-01 sin (t) in roll and pitch axes
simulated	-0.0057228544	0.8094600020	0.8095413040	
true	0.8037960073	0.8151518338	0.8037960073	0.1 e-01 sin (t) in all thre axes
simulated	0.8037907780	0.8151516246	0.8037955006	

Table-2 : Several runs of mid-frame method with 2.5 mins alignment time

(under pure coning motion of 2 sin (2 π 0.25 t) in roll and pitch axes)

	$\delta \psi^o$	$\delta \phi^o$	$\delta \theta^o$
	0.11	-0.004	0.026
	0.209	-0.004	0.027
	-0.15	-0.004	**0.028**
	-0.06	-0.005	0.024
	0.04	-0.004	0.025
	0.05	-0.004	0.025
	-0.001	-0.002	0.027
	-0.012	**-0.006**	0.023
	0.16	-0.005	0.025
	0.23	-0.005	0.024
	-0.06	-0.004	0.022
	0.36	-0.004	0.022
	0.36	-0.005	0.023
	0.08	-0.006	0.023
	0.25	-0.006	0.024
	0.19	-0.006	0.024
	0.13	-0.005	0.022
	0.07	-0.005	0.024
	0.03	-0.006	0.024
	0.1	-0.005	0.023
Accuracy bounds	0.36	-0.006	0.028

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Appendix

Time Propagation of Attitude Transformation Matrix

Algorithms for attitude propagation for strapdown inertial navigation systems are treated at length in the paper by Savage [6]. The relevant final results and algorithms are taken from the mentioned reference for the construction of $C_{n_k}^{n_{k+l}}$ and $\hat{C}_{b_{k+l}}^{b_k}$ required for propagating the DCM $\hat{C}_b^n$ (k).

Body Frame Rotation

The $\hat{C}_{b_{k+l}}^{b_k}$ can be expressed in terms of a rotation vector ρ_m defining the frame b_{k+l} attitude relative to frame b_k, given by

$$\hat{C}_{b_{k+l}}^{b_k} = I + \frac{\sin(\rho_m)}{\rho_m}(\rho_m \times) + \frac{1-\cos(\rho_m)}{\rho_m^2}(\rho_m \times)(\rho_m \times) \tag{18}$$

The ρ is calculated as the integral from instant k to instant $k + l$ of the general ρ equation

$$\dot{\rho} = \omega_{ib}^b + \frac{1}{2}\rho \times \omega_{ib}^b + \frac{1}{\rho^2}\left(1 - \frac{\rho \sin\rho}{2(1-\cos\rho)}\right)\rho \times (\rho \times \omega_{ib}^b) \tag{19}$$

Eqn.19 is well known in literature as the Bortz equation. Inertial angular rate ω_{ib}^b, is direct measurement from the strapdown angular rate sensors.

The attitude rotation vector is then obtained ρ_m is then obtained as the integral of Eqn. 19 from time k to time k + l

$$\rho(t) = \int_{t_k}^{t} \dot{\rho}(\tau)\, dt \qquad \rho_m = \rho(t_{k+l}) \tag{20}$$

A recursive algorithm for the computation of ρ_m is obtained after suitable approximations to the generic integral equation of $\dot{\rho}$ that includes non-cummutative part of the Bortz Equation. Given an SDINS gyros readout $\theta_i = \int \omega_{ib}^b \, dt$ of body incremental angles at any instant $k \le i < l$, the overall digital algorithm is as follows :

$\alpha = 0 \,;\, \beta = 0 \,;$

$i = k \,;$

Looping while

$(i < k + l)$

$\delta\beta = \frac{1}{2}(\alpha + \frac{1}{6}\theta_i - 1) \times \theta_i \,;$

$\beta = \beta^2 + \delta\beta \,;$

$\alpha = \alpha + \theta_i \,;$

$i = i + 1 \,;$

continue

$\rho_m = \alpha + \beta \,;$

The ρ_m through the above recursive procedure is substituted in Rodrigues formula given in Eqn. (18) to obtain transformation due to body roation $\hat{C}_{b_{k+l}}^{b_k}$.

Navigation Frame Rotation

The $C_{n_k}^{n_{k+l}}$ is the transformation matrix arising due to navigation frame rotation with respect to the inertial frame described by the angular rate vector ω_{in}^n the ω_{in}^n is composed of angular rate of Earth fixed frame relative to inertial frame ω_{ie}^n and angular rate of navigation frame with respect to Earth fixed frame ω_{en}^n. For the case of SDINS on oscillatory base, the latitude and longitude does not change, hence the angular rate of navigation frame relative to Earth fixed frame $\omega_{en}^n = 0$.

Following similar argument as for the body frame rotation, the transformation in the navigation frame arising due to Earth rotation could be expressed as

$$C_{n_k}^{n_{k+l}} = I + \frac{\sin(\zeta_n)}{\zeta_n}(\zeta_n \times) + \frac{1-\cos(\zeta_n)}{\zeta_n^2}(\zeta_n \times)(\zeta_n \times) \qquad (21)$$

where ζ_n is the rotation vector defining the frame n_{k+l} attitude relative to frame n_k..

As the number of sample between instant k and instant $k + l$ is kept small, typically four, the magnitude of ζ_n is very small. And because ω_{ie}^n is slowly changing over a typical update cycle, the non-commutative parts of the Bortz Equation can be neglected without loss of accuracy. The simplified form for evaluation of ζ_n is given by

$$\zeta(t) \approx \int_{t_k}^{t} \omega_{ie}^n \, dt \qquad \zeta_n = \zeta(t_{k+l}) \qquad (22)$$

And since $\| \zeta_n \|$ is small, the higher powers in the taylor series expansion of the terms $\frac{\sin(\zeta_n)}{\zeta_n}$ and $\frac{1-\cos(\zeta_n)}{\zeta_n^2}$ in the Eqn. (21) can be done away with leaving the fractions $\frac{\sin(\zeta_n)}{\zeta_n} \approx 1$ and $\frac{1-\cos(\zeta_n)}{\zeta_n^2} \approx \frac{1}{2}$. The simplified expression for $C_{n_k}^{n_{k+l}}$ accurate to the second order of ζ_n is obtained as

$$C_{n_k}^{n_{k+l}} = I - (\zeta_n \times) + \frac{1}{2}(\zeta_n \times)(\zeta_n \times) \qquad (23)$$

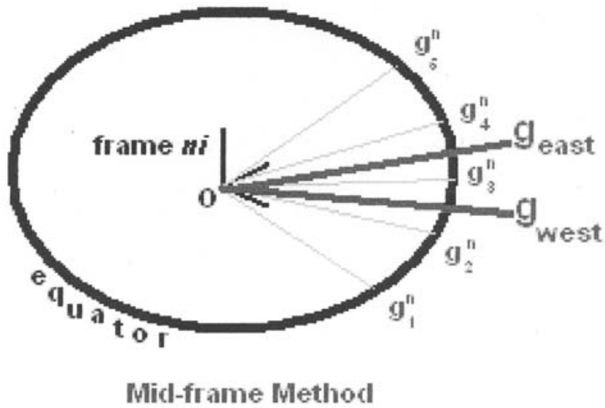

Fig.1 Pictorial summary of coarse alignment scheme for oscillatory base