A SIMPLE STRAIN RECOVERY BASED A-POSTERIORI ERROR ESTIMATOR FOR LAMINATED COMPOSITE PLATES

P.M. Mohite* and C.S. Upadhyay**

Abstract

A globally and locally reliable strain recovery based a-posteriori error estimator for laminated composite plates is proposed in this study. An extensive check of the local and global quality of the proposed error estimator is carried out for the bending problem. Effect of material orientation, ply stacking sequence, boundary condition and loading profile is carried out for a square plate. It is found that this estimator is reliable for almost all the cases, and even predicts errors for the preasymptotic range reliably.

***Key words**: Laminated Composite; plates; finite element; strain recovery; a-posteriori error*

Nomenclature

d	= plate thickness
p	= approximation order
$\mathbf{u}$	= generalized displacement field
x,y,z	= generalized coordinates
NEL	= number of elements
$NLAY$	= number of layers in laminate
U	= strain energy
ε	= strain vector
ε^*	= recovered strain vector
σ	= stress vector
τ	= element of interest
η_τ	= element error indicator for element τ
ξ_Ω	= global error indicator
ω	= local region
Ω	= global region

Introduction

Laminated composite plates are fast replacing metal alloys in most light transport vehicles. Increasingly, many aerospace and high speed rail components are being fabricated with composites. As the use of these materials grows, the need for refined analysis tools for composite structures assumes more importance. Several higher order plate models have been proposed in the literature (see [1-2] and references therein). In order to obtain local and global response quantities (e.g. first ply failure load, point of delamination, stress concentration at cut-outs, buckling load, etc.) a detailed finite element analysis of the laminated plate is required. In order to assure the reliability of the response quantities obtained from the finite element analysis, an adaptive analysis is required such that the discretisation error is controlled within acceptable tolerances. The adaptive analysis is driven by an a-posteriori error estimator. Several error estimators have been proposed in the literature (see [3-13] and references therein) for either the steady-state heat conduction, planar elasticity or Stokes problems. For laminated composite plates, not much has been reported in the literature (see [14] for an example).

Several possible projection based a-posteriori error estimators have been proposed in [15]. One of the possible projection proposed was based on strain recovery. In this paper, we define and study in detail the strain recovery based a-posteriori error estimator, as this estimator proved to be the most reliable of all the estimators proposed in [15]. The goal of the study is to present a reliable a-posteriori error estimator for laminated composite plates, with a further aim to make the definition of the estimator model independent. The estimator will be subjected to a series of tests, to ensure both local and global quality for a wide class of boundary conditions, material orientation and ply stacking sequences. An effort will be made to clearly bring out the effect of each factor contributing to the quality of the proposed error estimator. Further, the study will also bring out the effect of locking (of the plate, model) on the local quality of this error estimator.

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Model Problem

As the model problem we consider a laminated composite plate with the following higher order displacement field in terms of the thickness variables z (see [1]) :

$$\{\mathbf{u}\}(x,y,z)=\begin{Bmatrix}u(x,y,z)\\v(x,y,z)\\w(x,y,z)\end{Bmatrix}=\begin{Bmatrix}u_0(x,y)\\v_0(x,y)\\w_0(x,y)\end{Bmatrix}$$

$$+z\begin{Bmatrix}u_1(x,y)\\v_1(x,y)\\0\end{Bmatrix}+z^3\begin{Bmatrix}u_2(x,y)\\v_2(x,y)\\0\end{Bmatrix} \tag{1}$$

The higher order plate theory given by Eqn.(1), is one of the many proposed in the literature. We are taking this model in order to elucidate various principles associated with the design of strain-recovery based error estimators.

The laminate is made by stacking laminae with given material properties, orientation and ply thickness. For a given lamina 'l', the generalised Hooke's law (see [16-17]) gives :

$$\sigma^{(l)}(\mathbf{u}) = \overline{\mathbf{Q}}^{(l)}\,\varepsilon^{(l)}(\mathbf{u}) \tag{2}$$

Where,

$$\sigma^{l}(\mathbf{u})=\left\{\sigma^{(l)}_{xx}(\mathbf{u}),\sigma^{(l)}_{yy}(\mathbf{u}),\sigma^{(l)}_{zz}(\mathbf{u}),\sigma^{(l)}_{yz}(\mathbf{u}),\sigma^{(l)}_{xz}(\mathbf{u}),\sigma^{(l)}_{xy}(\mathbf{u})\right\}^T$$

is the enginering stress vector for the l^{th} lamina;

$$\varepsilon^{l}(\mathbf{u})=\left\{\varepsilon^{(l)}_{xx}(\mathbf{u}),\varepsilon^{(l)}_{yy}(\mathbf{u}),\varepsilon^{(l)}_{zz}(\mathbf{u}),\gamma^{(l)}_{yz}(\mathbf{u}),\gamma^{(l)}_{xz}(\mathbf{u}),\gamma^{(l)}_{xy}(\mathbf{u})\right\}^T$$

is the engineering strain vector for the l^{th} lamina; $\overline{\mathbf{Q}}^{(l)}$ is the material matrixd for the l^{th} lamina (transformed to the x-y coordinate system). Using the definition of $\varepsilon^{(l)}$ in terms of the displacement field $\mathbf{u}\,(x, y, z)$, the strain energy $\mathcal{U}$, and the potential $\mathcal{V}$ due to the transverse loads acting on the top and bottom faces of the plate are :

$$\mathcal{U}(\mathbf{u})=\frac{1}{2}\int_{\Omega}\sigma(u)\,.\;\varepsilon(\mathbf{u})\,dV$$

$$=\frac{1}{2}\int_{\Omega\,2D}\sum_{i=1}^{NLAY}\int_{z_{i-1}}^{z_i}\left(\varepsilon^{(l)}(\mathbf{u})\right).\,\varepsilon(\mathbf{u})\,dz)\,dA$$

$$\mathcal{V}(\mathbf{u}) = \int_{R+} q^{+}\,w_0\,dA + \int_{R-} q^{-}\,w_0\,dA \tag{3}$$

Here Ω is the plate domain of interest (of rectangular cross section $\Omega_{2D}=\{(x,y)\,|\,0\le x\le a,\,0\le y\le b\}$ and depth d), as shown in Fig.1a; $NLAY$ is the number of laminae; z_i are the z coordinates of interlaminar interfaces (as shown in Fig.1b; R^+ and R^- are the top and bottom faces of the plate, respectively; $q^+(x,y)$ and $q^-(x,y)$ are the transverse loads on R^+ and R^-, respectively.

Thus, the total potential energy is given as :

$$\Pi(\mathbf{u}) = \mathcal{U}(\mathbf{u}) - \mathcal{V}(\mathbf{u}) \tag{4}$$

Minimizing Π ($\mathbf{u}$) with repesct to $\mathbf{u}$, we get,

$$\beta(\mathbf{u},\delta\mathbf{u})=\int_{\Omega\,2D}\sum_{i=1}^{NLAY}\int_{z_{i-1}}^{z_i}\left(\sigma^{(l)}(\mathbf{u})\right).\,\varepsilon(d\,\mathbf{u})\,dz)\,dA$$

$$\mathcal{F}(\delta\mathbf{u}) = \int_{R+} q^{+}\,\delta w_0\,dA + \int_{R-} q^{-}\,\delta w_0\,dA \tag{5}$$

The variational formulation (5) is often written in its equivalent form in terms of stress resultants at the central surface Ω_{2D} of the plate. The above formulation leads to seven coupled equation in terms of the seven independent functions u_0, v_0, w_0, u_1, v_1, u_2, v_2. The finite element formulation of the above problem follows by replacing the given functions by the approximating series representation in terms of the basis functions. Here we will take elementwise p order approximation for all the unknown functions.

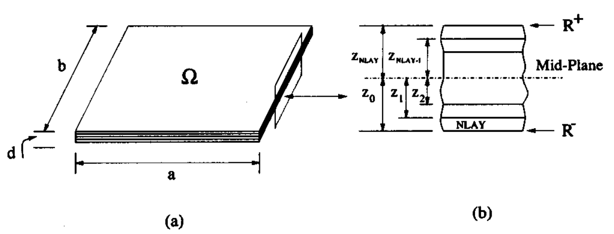

Fig.1 Plate domain with interlaminar interfaces and top and bottom faces

Strain Recovery Procedure

The finite element solution leads to a stress field that is less accurate than the displacement field itself. Zienkiewicz-Zhu [8] gave a stress recovery procedure for the extraction of a more accurate stress field, using a patch projection of the finite element data. This was extended to isotropic plates (see [24]), where the stress resultants were recovered using a procedure similar to [8]. However, in [21-22] it has been observed that instead of nodal patches for projection, use of element patches, i.e. by using one or two layers of elements surrounding an element, a better stress recovery can be obtained. Below, we present the proposed strain I recovery procedure (see [15] as well), which is based on recovery of strain components instead of stress resultants.

Following the representation of the solution by Eqn.(1), we get the components of strain as:

$$\varepsilon = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^{0}_{xx} \\ \varepsilon^{0}_{yy} \\ \gamma^{0}_{yz} \\ \gamma^{0}_{xz} \\ \gamma^{0}_{xy} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon^{1}_{xx} \\ \varepsilon^{1}_{yy} \\ 0 \\ 0 \\ \gamma^{1}_{xy} \end{Bmatrix} + z^2 \begin{Bmatrix} 0 \\ 0 \\ \gamma^{1}_{yz} \\ \gamma^{1}_{xz} \\ 0 \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon^{2}_{xx} \\ \varepsilon^{2}_{yy} \\ 0 \\ 0 \\ \gamma^{2}_{xy} \end{Bmatrix} \tag{6}$$

The recovered strain ε^* is also assumed to have the same form (in terms of z) as the exact one. Thus, the recovered strain is also represented as :

$$\varepsilon^* = \begin{Bmatrix} \varepsilon^{*}_{xx} \\ \varepsilon^{*}_{yy} \\ \gamma^{*}_{yz} \\ \gamma^{*}_{xz} \\ \gamma^{*}_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^{*,0}_{xx} \\ \varepsilon^{*,0}_{yy} \\ \gamma^{*,0}_{yz} \\ \gamma^{*,0}_{xz} \\ \gamma^{*,0}_{xy} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon^{*,1}_{xx} \\ \varepsilon^{*,1}_{yy} \\ 0 \\ 0 \\ \gamma^{*,1}_{xy} \end{Bmatrix} + z^2 \begin{Bmatrix} 0 \\ 0 \\ \gamma^{*,1}_{yz} \\ \gamma^{*,1}_{xz} \\ 0 \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon^{*,2}_{xx} \\ \varepsilon^{*,2}_{yy} \\ 0 \\ 0 \\ \gamma^{*,2}_{xy} \end{Bmatrix} \tag{7}$$

Given the representation of ε^*, it is now desired to obtain the recovered strain field as a polynomial element by element, such that the recovered strain components are polynomials that are one order higher than the corresponding finite element strain components. Thus, if elements of order p are employed all the recovered in-plane strain components are polynomials of degree p and the out of plane strain components are polynomials of degree p+1, and are given by:

$$\varepsilon^{*,i}_{xx} = \sum_{j=1}^{NIN} \varepsilon^{*,i}_{xx,j}\; q_j(\hat{x},\hat{y}) \quad i = 0, 1, 2$$

$$\varepsilon^{*,i}_{yy} = \sum_{j=1}^{NIN} \varepsilon^{*,i}_{yy,j}\; q_j(\hat{x},\hat{y}) \quad i = 0, 1, 2$$

$$\gamma^{*,i}_{yz} = \sum_{j=1}^{NOUT} \gamma^{*,i}_{yz,j}\; q_j(\hat{x},\hat{y}) \quad i = 0, 1$$

$$\gamma^{*,i}_{xz} = \sum_{j=1}^{NOUT} \gamma^{*,i}_{xz,j}\; q_j(\hat{x},\hat{y}) \quad i = 0, 1$$

$$\gamma^{*,i}_{xy} = \sum_{j=1}^{NIN} \gamma^{*,i}_{xy,j}\; q_j(\hat{x},\hat{y}) \quad i = 0, 1, 2 \tag{8}$$

where $NIN = (p+1)(p+2)/2$, $NOUT = (p+2)(p+3)/2$ and $q_j(\hat{x},\hat{y})$ are the monomials given by :

$$q_1(\hat{x},\hat{y}) = 1,\ \ q_2(\hat{x},\hat{y}) = \hat{x},\ \ q_3(\hat{x},\hat{y}) = \hat{y},$$

$$q_4(\hat{x},\hat{y}) = \hat{x}^2,\ \ q_5(\hat{x},\hat{y}) = \hat{x}\,\hat{y},\ \ q_6(\hat{x},\hat{y}) = \hat{y}^2 \tag{9}$$

Here $\hat{x} = x - x^{\tau}_{c}$, $\hat{y} = y - y^{\tau}_{c}$ are the local coordinates with the origin at the centroid of the element of interest, τ.

The unknown coefficients of the strain components are obtained using a local energy .f projection of the finite element strains, over a layer of elements surrounding element τ (see Fig.2). This is done by finding the coefficients that minimize

$$J = \frac{1}{2}\sum_{i=0}^{N_{P-1}} \int_{\tau^P_i} \left(\sum_{l=1}^{NLAY} \int_{z_{l-1}}^{z_l} \left(\left(\varepsilon^* - \varepsilon_{FE}\right) . \overline{Q}^{(l)} (\varepsilon - \varepsilon) \right) dz \right) dA \tag{10}$$

where N_p is the number of elements in the patch P, and $NLAY$ is the number of laminae. Note that J is the strain

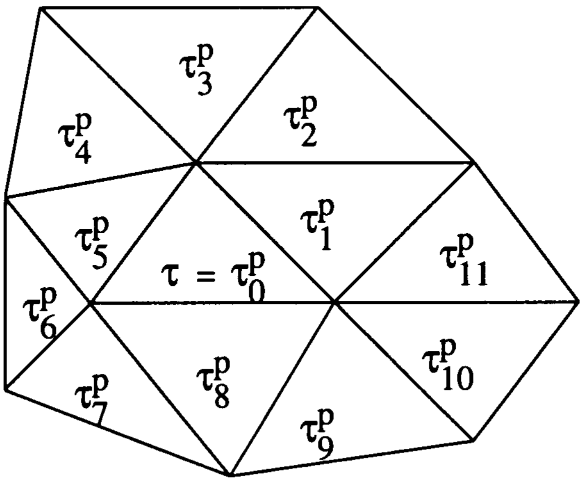

Fig.2 An element τ with a layer of elements $\{\tau_i^P\}_{i=0}^{11}$ *surrounding it, forming the patch P*

energy of the error in the strain, $\varepsilon^* - \varepsilon_{FE}$, where ε_{FE} is the finite element strain.

The minimization of J gives as many linearly independent equations as the number of coefficients in Eqn.(10). The coefficients are solved for each element patch P, and the values retained for the element τ.

Remark 1 : No attempt is made here to obtain a smoothened stress or strain field (as prescribed in [9-10]). It is expected, following [21], [23], that the recovered strain field will be more accurate than that obtained by the finite element solution.

Remark 2 (Presence of Material interfaces) : If the plate is formed by using different materials in subregions (Fig.3), then for strain recovery only elements lying in the same material region will be used for the projection. For example, given the element τ_0^P, the patch consists of elements $\{\tau_i^P\}_{i=0}^{6}$.

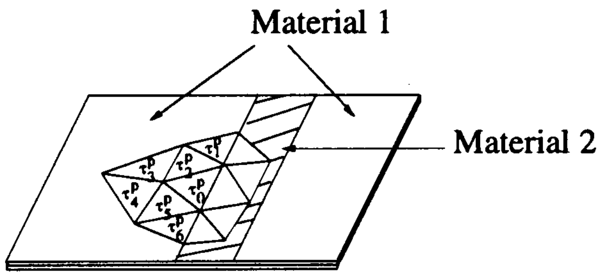

Fig.3 Rectangular plate showing the presence of material interface

Definition of a-Posteriori Error Estimator Based on Strain Recovery

The recovered strain ε^* can be used to define an a-posteriori estimate of the error. The element error indicator η_τ, for an element τ is given as:

$$\eta_\tau^2 = \int_\tau \left(\sum_{l=1}^{NLAY} \int_{z_{l-1}}^{z_i} \left(\left(\varepsilon^* - \varepsilon_{FE}\right) . \overline{Q}^i \left(\varepsilon^* - \varepsilon_{FE}\right) \right) dz \right) dA \tag{11}$$

The element error indicators can be used to define the global error estimator ξ_Ω as :

$$\xi_\Omega = \sqrt{\sum_{\tau-1}^{NEL} \eta_\tau^2} \tag{12}$$

where NEL is the total number of elements in the mesh.

The error estimator based on the recovered strain has to be tested for robustness and accuracy. Following [21, [23], it is imperative to subject an estimator to rigorous bench marking tests in order to ascertain the quality of the estimator for the class of materials, domains, loading and boundary conditions of interest. In [20], [23], a rigorous mathematical proof was given, which lead to a simple computer-based procedure for testing the quality of a-posteriori error estimators for general second order elliptic problems. The basic idea of [20] can be outlined as follows:

Let $\overline{\omega}$ be a small subregion of interest, lying inside the domain Ω. Then asymptotically, for $\overline{\omega}$ sufficiently small, the finite element solution is essentially the best approximation of the local $(p+1)^{th}$ order Taylor series expansion of the exact solution **u**, over a region slightly bigger than $\overline{\omega}$. The assumptions for the asymptotic error analysis were:

- All global contributions to error in the local region $\overline{\omega}$ (i.e. pollution error) were negligible.
- The dominant part of the local error was due to the $(p+1)^{th}$ degree terms of the local Taylor series expansion of the exact solution.

However, for laminated composite plates no such detailed interior analysis of local error exists. Following [25], [28], we get the global component of the error (for a rectangular plate) in a local region $\overline{\omega}$ due to only boundary layer effect. The effect of the thickness of the plate, d, on the convergence rate is seen through a slowing down in the setting of asymptotic behavior, i.e. a more refined mesh may be required to get asymptotic behavior (this is also known as locking). For the h-version of the finite element method, the boundary layer effect can be controlled by using sufficient mesh refinements near the boundaries. Assuming that the thickness d is fixed (away from zero) we get, for the error $\mathbf{e} = \mathbf{u} - \mathbf{u}_{FE}$,

$$\|\mathbf{e}\|_{E(\Omega)} \leq C(d)\, h^{\mu} \tag{13}$$

where $\mu = \min(p, r)$ and τ depends on the regularity of the solution $\mathbf{u}$ to the plate model;

$\|\cdot\|_{E(\overline{\omega})}$ is the energy norm given by $\|\mathbf{u}\|_{E(\overline{\omega})} = \sqrt{2\,\mathcal{U}_{\overline{\omega}}(\mathbf{u})}$ where $\mathcal{U}_{\overline{\omega}}$ is the strain energy of $\mathbf{u}$ over region $\overline{\omega}$.

We further assume that, for a subregion $(\overline{\omega}) \in \Omega$, sufficiently removed from the boundary

$$\|\mathbf{e}\|_{E(\overline{\omega})} \leq C(d)\, h^{p} \tag{14}$$

in the absence of boundary layer effects, and for a fixed d (see [18] for a detailed proof on convergence of local error for isotropic plates). Thus, if the finite element solution is obtained over the same mesh using (p+1) order elements, the error $\mathbf{e}^{p+1} = \mathbf{u} - \mathbf{u}_{FE}^{p+1}$ in the finite element solution $\mathbf{u}_{FE}^{p+1}$ satisfies :

$$\|\mathbf{e}^{p+1}\|_{E(\overline{\omega})} \leq C(d)\, h^{p+1} \tag{15}$$

Hence, we can obtain,

$$\begin{aligned} \|\mathbf{e}\|_{E(\overline{\omega})} &= \|\mathbf{u} - \mathbf{u}_{FE}^{p+1} + \mathbf{u}_{FE}^{p+1} - \mathbf{u}_{FE}\|_{E(\overline{\omega})} \leq \|\mathbf{u} - \mathbf{u}_{FE}^{p+1}\|_{E(\overline{\omega})} \\ &+ \|\mathbf{u}_{FE}^{p+1} - \mathbf{u}_{FE}\|_{E(\overline{\omega})} \approx \|\mathbf{u}_{FE}^{p+1} - \mathbf{u}_{FE}\|_{E(\overline{\omega})} \end{aligned} \tag{16}$$

or the error is essentially the difference between the $(p+1)$ order solution and the p order solution, when $h \to 0$.

Letting $\xi_{\overline{\omega}}$ be the error estimator for subregion $\overline{\omega}$ we define

$$\kappa_{\overline{\omega}} = \frac{\xi_{\overline{\omega}}}{\|\mathbf{e}\|_{E(\overline{\omega})}} \approx \frac{\xi_{\overline{\omega}}}{\|\mathbf{u}_{FE}^{p+1} - \mathbf{u}_{FE}\|_{E(\overline{\omega})}} \tag{17}$$

where $\kappa_{\overline{\omega}}$ is the effectivity index for the subregion $\overline{\omega}$. Ideally $\kappa_{\overline{\omega}} = 1$ is desired. However, we can say that the estimator is reliable if $0.8 \leq \kappa_{\overline{\omega}} \leq 1.2$ (heuristic choice). Defining

$$\Re_{\overline{\omega}} = |1 - \kappa_{\overline{\omega}}| + |1 - \frac{1}{\kappa_{\overline{\omega}}}| \tag{18}$$

as the *robustness measure* for the error estimator, we can say that an estimator is robust if $\Re_{\overline{\omega}} = 0.45$ (corresponding to the reliable $\kappa_{\overline{\omega}}$ fixed above).

Remark 3 : An estimator may overestimate or underestimate the error, depending on the region of interest, mesh, loading data and boundary conditions. The robustness measure gives a uniform scale for measuring the reliability over a given domain.

Remark 4 : If the solution has converged for the given p, the error $\|\mathbf{u}_{FE}^{p+1} - \mathbf{u}_{FE}\|_{E(\overline{\omega})}$ will be close to zero. In this case, the effectivity index will not be meaningful. In such cases, if $\varepsilon^{*} \approx \varepsilon_{FE}$ then the recovery is said to be reliable.

Remark 5 : In case of locking the rate of convergence becomes sub-optimal. In such a situation no error estimator will be reliable for the boundary patches as well as for the interior patches. However, if the locking is taken care of (by proper h or p refinement, or by selecting locking free plate models) the estimator performs well for interior as well as, boundary patches (see [20]). In this study, we will bring out the effect of locking by using low p (p=2) or high p (p=3), especially for thin plates.

The effectivity index κ_{Ω} is important in giving the stopping criterion for an adaptive mesh refinement procedure accurately. Small $\kappa_{\overline{\omega}}$ (for subregion $\overline{\omega}$) is essential to ensure that the right elements get refined. Thus for elements with significant error, the estimator should be reliable.

Validation of Quality of the a-Posteriori Error Estimator

The computational analysis of laminated composite plates requires consideration of several factors. The major factors are:

- Effect of ply orientation, stacking sequence
- Effect of boundary conditions
- Effect of type of loading
- Effect of thickness of plate
- Effect of plate model
- Effect of approximation order

In this study, we are fixing the plate model to the one given by Eqn. (1). For this plate model, the quality of the recovery based a-posteriori error estimator will be studied. The properties of the material of interest (T300/5208 Graphite/Epoxy (prepreg)) are given as [29} :

$E_u = 132.5$ GPa; $E_{tt} = 10.8$ GPa; $\nu_{lt} = 0.24$;
$\nu_{tt} = 0.49$; $G_{lt} = 5.7$ GPa

Here, the analysis is done for two types of boundary conditions, namely, simply supported and clamped. When we specify any boundary condition it is imposed on all four edges. By simply supported we mean soft simply supported in which transverse displacement and displacement tangential to the edge is fixed (eg. for edge x=constant, $v = w = 0$). For clamped boundary condition all the three displacements are fixed (eg. for any edge $u = v = w = 0$).

Below, we present a detailed analysis of the global and local quality of the proposed error estimator.

Case 1: Quality for elements in the interior of the domain, $\Omega_{2D\ ,interior}$

The local performance of the error estimator has to be understood for the various possible scenarios separately. We know that the boundary layer effect in the finite element solution may be present only in few layers of elements adjoining the boundary. For elements removed from boundary, the finite element solution behaves like the local best approximation, i.e. the local error converges at the optimal rate. Thus, the first check for the quality of the error estimator should be for elements in the interior of the domain, i.e. for elements in the subregion $\Omega_{2D,\ interior}$ In Fig.4, the elements at the boundary and interior for the meshes used, are shown.

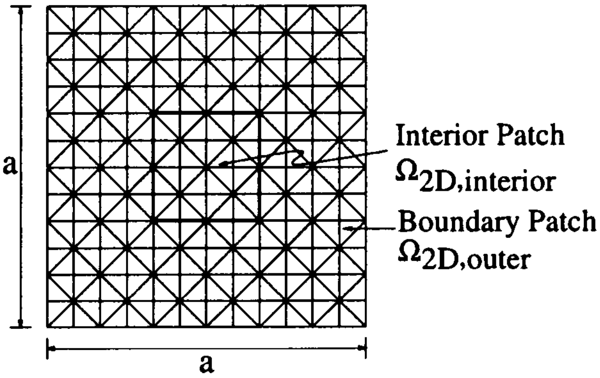

Fig.4 : The square plate of dimensions a x a, with the mesh shown. The inner and outer patches of elements are given by $\Omega_{2D,\ interior}$ and $\Omega_{2D,\ outer}$, respectively

All the numerical results presented will be for the four-layered laminates, with the ply thickness fixed to $d_l = 0.127$ mm, $l = 1,2,3,4$. In Tables-1 to 5, the values of the effectivity index, for the elements in the interior of the domain (see Fig.4) are given for various ply orientations and boundary conditions. Effect of plate thickness is accounted for by taking $\frac{a}{d} = 5$ (thick plate), 10 (moderately thick plate), 100 (thin plate). Elements of order p=2 are taken for all the problems. The transverse load is fixed to a uniformly distributed load of intensity $q^+ = 2N/mm^2$ and $q^- = 0$.

From the results we observe that:

- For all the ply orientations and stacking sequences, the estimator is very reliable for thick and moderately thick plates, i.e. $\Re_\tau \leq 0.25$.
- For the thin plates the elemental quality deteriorates. However, for the group of elements in the interior $\Re_{2D,interior} \leq 0.2$.
- The results are relatively independent of the boundary condition type.

For the thin plates $\left(\frac{a}{d} = 100\right)$, the error estimator was found to be globally accurate ($0.92 \leq \xi_\Omega \leq 1.03$), and the global error was found to be greater than 10%. This indicates a tendency of the model to lock slightly. Thus, the results of Tables-1 and 2 are not asymptotic in nature.

Table-1 : Quality of error estimator for elements in $\Omega_{2D,\ interior}$: p = 2, uniform transverse load ($q^+ = 2N/mm^2$), $[0/90]_s$ laminate

$\frac{a}{d}$ ratio	Simply supported		Clamped	
	ξ_{max}	ξ_{min}	ξ_{max}	ξ_{min}
5	1.116	0.986	1.142	1.004
10	1.181	0.980	1.152	0.989
100	1.646	0.757	1.633	0.710

Table-2 : Quality of error estimator for elements in $\Omega_{2D,\ interior}$: p = 2, uniform transverse load ($q^+ = 2N/mm^2$), $[45/-45]_s$ laminate

$\frac{a}{d}$ ratio	Simply supported		Clamped	
	ξ_{max}	ξ_{min}	ξ_{max}	ξ_{min}
5	1.082	0.964	1.056	0.976
10	1.043	0.938	1.089	0.963
100	1.578	0.614	1.219	0.661

Table-3 : Quality of error estimator for elements in $\Omega_{2D,\ interior}$: p = 2, uniform transverse load ($q^+ = 2N/mm^2$), [0/90/0/90] laminate

$\frac{a}{d}$ ratio	Simply supported		Clamped	
	ξ_{max}	ξ_{min}	ξ_{max}	ξ_{min}
5	1.019	0.994	1.038	0.984
10	1.018	0.993	1.055	0.997

Table-4 : Quality of error estimator for elements in $\Omega_{2D,\ interior}$: p = 2, uniform transverse load ($q^+ = 2N/mm^2$), [0/45/90/45] laminate

$\frac{a}{d}$ ratio	Simply supported		Clamped	
	ξ_{max}	ξ_{min}	ξ_{max}	ξ_{min}
5	1.127	0.907	1.063	0.973
10	1.112	0.928	1.059	0.978

To get the asymptotic results, p=3 was taken to solve the problem. The result is given in Table-5. From Table-5 we note that:

- In the absence of locking, the error estimator is reliable $\left(\Re_\tau \le 0.05\right)$.

Table-5 : Quality of error estimator for elements in $\Omega_{2D,\ interior}$ for thin plates ($\frac{a}{d}$ = 100) : Unlocked solution with p=3, uniform transverse load ($q^+ = 2N/mm^2$), for [0/90]s and $[45/-45]_s$ laminates. The values in parenthesis are the preasymptotic ones (or "locked")

B.C. Type	$[0/90]_s$		$[45/-45]_s$	
	ξ_{max}	ξ_{min}	ξ_{max}	ξ_{min}
Simply supported	1.041 (1.646)	0.984 (0.757)	1.052 (1.578)	0.881 (0.614)
Clamped	1.099 (1.633)	0.995 (0.710)	1.091 (1.219)	0.948 (0.661)

- The error estimator is relatively insensitive to the thickness of the plate, when no locking effects are present.
- Even when locking is present, the performance of the estimator for the interior patch is very good, i.e. $\Re_{2D,interior} \le 0.15$.

In general, we can conclude that the proposed estimator is very reliable for interior patches of elements.

Case 2: Quality for elements in the boundary patch, $\Omega_{2D,\ outer}$

Following [23], the quality of error estimators for elements at or near the domain boundary is expected to be different from that for the elements in the interior of the domain. This is because of the boundary layer in the exact solution as well as in the finite element solution. For the laminated plates, this effect can be significant. Thus, the local quality of the proposed error estimator is investigated separately for elements in $\Omega_{2D,\ outer}$. From the results given in Tables-6 to 10, we observe that:

- For the simply-supported plate, the estimator is reliable upto the boundary for all $\frac{a}{d}$ ratios and symmetric cross ply ($[0/90]_s$), as well as the antisymmetric cross ply ([0/90/0/90]), with $\Re_\tau \le 0.4$.
- For the angle ply laminate ($[45/-45]_s$) the estimator is not reliable for the simply supported plate.
- For the clamped plate, the estimator is not reliable (elementwise) for all lamination sequences.

We note that the elementwise quality of the error estimator cannot be guaranteed for elements at the boundary, especially for angle ply laminates. However, it was found that $\Re_{\Omega_{2D,outer}} \leq 0.2$ for all the cases considered. Hence, for a group (or patch) of elements, the estimator is very reliable, even for the locked case.

Table-6 : Quality of error estimator for elements in $\Omega_{2D,\,outer}$: p = 2, uniform transverse load ($q^+ = 2N/mm^2$), $[0/90]_s$ laminate

$\frac{a}{d}$ ratio	Simply supported		Clamped	
	ξ_{max}	ξ_{min}	ξ_{max}	ξ_{min}
5	1.134	0.925	1.965	0.641
10	1.115	0.874	2.138	0.641
100	1.313	0.853	1.295	0.729

Table-7 : Quality of error estimator for elements in $\Omega_{2D,\,outer}$: p = 2, uniform transverse load ($q^+ = 2N/mm^2$), $[45/\text{-}45]_s$ laminate

$\frac{a}{d}$ ratio	Simply supported		Clamped	
	ξ_{max}	ξ_{min}	ξ_{max}	ξ_{min}
5	2.464	0.659	2.057	0.571
10	2.085	0.626	1.888	0.569
100	1.419	0.597	1.520	0.577

Table-8 : Quality of error estimator for elements in $\Omega_{2D,\,outer}$: p = 2, uniform transverse load ($q^+ = 2N/mm^2$), [0/90/0/90] laminate

$\frac{a}{d}$ ratio	Simply supported		Clamped	
	ξ_{max}	ξ_{min}	ξ_{max}	ξ_{min}
5	1.099	0.959	1.706	0.687
10	1.088	0.950	1.649	0.653

Table-9 : Quality of error estimator for elements in $\Omega_{2D,\,outer}$: p = 2, uniform transverse load ($q^+ = 2N/mm^2$), [0/45/90/45] laminate

$\frac{a}{d}$ ratio	Simply supported		Clamped	
	ξ_{max}	ξ_{min}	ξ_{max}	ξ_{min}
5	2.052	0.651	1.096	0.972
10	1.687	0.604	1.594	0.649

Remark 6 : It is observed from the results that in neighbouring elements (see Fig.5), the value of ξ_τ seems to alternate between a high and low value. This is the so-called "chattering" effect observed in [19].

Remark 7: The estimator does remarkably well in estimating the global error, and hence should lead to an accurate stopping criterion for an adaptive mesh refinement process.

Case 3: Influence of transverse loading type and mesh topology

The local solution depends on the transverse loading acting on the plate. In order to see the effect of the load type on the quality of error estimator, we let q+ = 10 sin $(\frac{\pi x}{a})$ sin $(\frac{\pi y}{a})$ N/mm^2 be the intensity of the applied load.

Table-10 : Quality of error estimator for elements in $\Omega_{2D,\,outer}$ for thin plates ($\frac{a}{d}$ = 100) : Unlocked solution with p=3, uniform transverse load ($q^+ = 2N/mm^2$), for [0/90]s and $[45/\text{-}45]_s$ laminates. The values in parenthesis are the preasymptotic ones (or "locked")

B.C. Type	$[0/90]_s$		$[45/\text{-}45]_s$	
	ξ_{max}	ξ_{min}	ξ_{max}	ξ_{min}
Simply supported	1.225 (1.312)	0.892 (0.853)	1.649 (1.419)	0.649 (0.597
Clamped	1.375 (1.295)	0.777 (0.729)	1.347 (1.520)	0.776 (0.577)

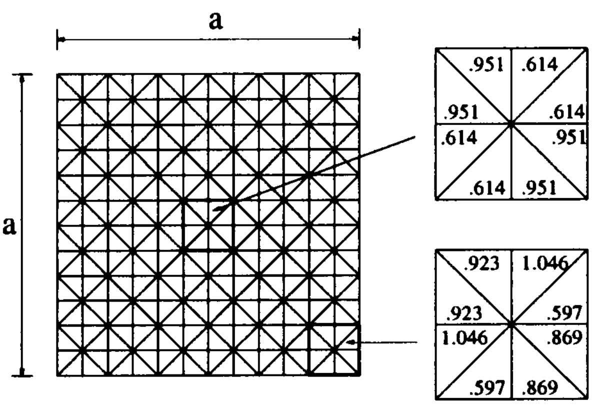

Fig.5 The chattering effect observed in $[45/\text{-}45]_s$ laminate, simply supported, uniform transversed load $q^+ = 2N/mm^2$, p = 2

Table-11 : Effect of change of loading on quality of error estimator : Range of effectivity index for $\Omega_{2D,\ interior}$ and $\Omega_{2D,\ outer}$. p=2. Sinusoidal loading $q^+ = 10 \sin(\frac{\pi x}{a}) \sin(\frac{\pi y}{a})$ N/mm^2, [0/90]$_s$ laminate

$\frac{a}{d}$ ratio	B.C. Type	$\Omega_{2D,\ interior}$		$\Omega_{2D,\ outer}$		Ω
		ξ_{max}	ξ_{min}	ξ_{max}	ξ_{min}	ξ_Ω
10	Simply supported	1.062	0.972	1.096	0.972	1.008
	Clamped	1.059	0.978	1.594	0.649	1.001

Table-12 : Effect of mesh topology on error estimator. Regular pattern p = 2, [0/90]$_s$ laminate, uniform transverse loading with $q^+ = 2N/mm2$. Range of ξ_τ for $\Omega_{2D,\ interior}$ and $\Omega_{2D,\ outer}$

$\frac{a}{d}$ ratio	B.C. Type	$\Omega_{2D,\ interior}$		$\Omega_{2D,\ outer}$		Ω
		ξ_{max}	ξ_{min}	ξ_{max}	ξ_{min}	ξ_Ω
5	Simply supported	1.025	0.909	1.073	0.903	0.997
	Clamped	1.035	0.954	2.011	0.481	0.998
10	Simply supported	1.023	0.893	1.064	0.893	0.984
	Clamped	1.062	0.919	2.158	0.438	0.933
100	Simply supported	1.203	0.855	1.027	0.704	0.961
	Clamped	1.241	0.874	1.097	0.414	0.931

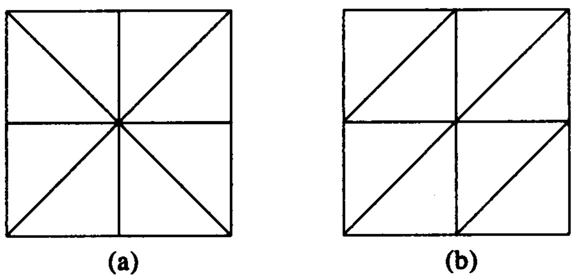

Fig.6 The periodic pattern (a) Union Jack (b) Regular

In Table-11, we report the values of the best and worst element effectivity indices obtained for $\Omega_{2D,\ interior}$ and $\Omega_{2D,\ outer}$. From the result we note for this loading the observations are similar to that obtained for a uniform load.

For triangular elements, the topology of the elements can affect quality of the error estimation (see [20] for details). In order to see the influence of the mesh topology, we consider meshes formed by repetition of the periodic patterns shown in Fig.6.

All the results given till now are for the meshes of Union Jack type. In Table-12, we give some values of ξ_{max} and ξ_{min} obtained for meshes of Regular type and the [0/90]$_s$ laminate. From the results we note that:

- For $\Omega_{2D,\ interior}$ the estimator is more accurate for the Regular pattern, as compared to the Union Jack pattern. Here $\Re_\tau \leq 0.45$ for all cases. Even for thin plate, this result is valid, even when the error is large (> 15%).
- For $\Omega_{2D,\ outer}$ the estimator behaves similarly for both patterns. Here, the elemental accuracy is not good ($0.4 \leq \xi_\tau \leq 2.2$) but the patch accuracy is very good $\Re_{patch} \leq 0.3$.

Conclusions

A simple, strain recovery based error estimator is proposed in this study. A detailed analysis of the local and global quality of this error estimator for laminated composite plate is carried out, when the plate is loaded by a distributed transverse load. From the study we can conclude that :

1. The error estimator is very reliable globally.

2. For mesh patches in the interior of the domain the estimator is very reliable.
3. For boundary patches, the estimator is reliable but the quality is inferior as compared to that in the interior of the domain.
4. The estimator is relatively insensitive to the material orientation or ply stacking sequence.
5. For thin plates, in the pre asymptotic range, the estimator does well to predict the global error accurately. The prediction of the local error in element patches is also reliable.
6. Mesh topology does not affect the performance of the error estimator significantly.
7. The estimator is independent of the plate model by design.

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