TECHNICAL NOTE

BEHAVIOUR OF SHEAR DEFORMABLE PLATE BENDING FINITE ELEMENTS FOR COMPOSITES

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Abstract

A thorough study of the behaviour of shear deformable plate bending finite elements is carried out to investigate their performance when applied to static and free vibration analyses of laminated composite plates. 4-noded, 8-noded and 9-noded quadrilateral isoparametric finite elements based on first-order shear deformation theory and a 4-noded element based on higher-order shear deformation theory are considered. The numerical results indicating the effect of order of integration on the accuracy of the results and convergence of transverse displacements and stresses are presented. The results indicate that a 4-noded element with seven degrees of freedom per node based on higher-order shear deformation theory is required to predict the deflection as well as the stresses accurately and a 16x 16 mesh division for full plate is necessary for obtaining converged results. For the case of free vibration analysis, the element based on first-order shear deformation theory is found to be sufficient.

Nomenclature

x, y, z	= Cartesian co-ordinates
u, v, w	= displacements in x, y, z directions
u_o, v_o, w_o	= mid-plane displacements in x, y, z directions
θ_x, θ_y	= total slopes in the x and y directions
σ_x, σ_y, τ_{xy}, τ_{xz}, τ_{yz}	= stress components
a	= length of the plate
b	= width of the plate
h	= total thickness of the plate
E_1, E_2	= Young's moduli along and transverse direction of the fibre
G_{12}, G_{13}, G_{23}	= in-plane and transverse shear moduli
ν_{12}	= in-plane Poisson's ratio
$\overline{w}$	= non-dimensional central deflection
$\overline{\sigma}_x$, $\overline{\sigma}_y$, $\overline{\tau}_{xy}$, $\overline{\tau}_{xz}$, $\overline{\tau}_{yz}$	= non-dimensional stresses
q	= intensity of load
$\overline{\omega}$	= non-dimensional fundamental frequency
ρ	= density of the material

Introduction

Analysis of laminated composite plates has been a subject of keen interest for structural engineers for quite a long time and continues to be so. Though a large number of approaches and a significant number of finite element models have been proposed by various researchers [1,2,3] for the analysis of such plates, further efforts seem to be necessary in order to fully investigate specific aspects of analysis. Behaviour of angle-ply laminates is different from that of cross-ply laminates. Symmetrically laminated plates may behave differently from antisymmetrically laminated plates. When one browses through literature, he finds 4-noded, 8-noded or 9-noded elements based on first-order, second-order or third-order shear deformation theories. Different authors recommend different orders of integration for obtaining stiffness and mass matrices. Many authors claim that very good results can be obtained by using a 4x4 mesh of finite elements. Sivakumaran et al. [1] used a 9-noded isoparametric element, based on three different displacement models, employing selective reduced integration scheme for the evaluation of element stiffness matrix and the results are reported to be converged with 4x4 mesh for full plate. Ghosh and Dey [2,4] reported that accurate results are obtained for both thin and thick plates with full integration, using a 4-noded element

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based on higher-order shear deformation theory. The results are reported to be converged with 8x8 mesh division for full plate. Kant et al. [3,5] conducted a study on 9-noded element based on higher-order shear deformation theory by employing selective reduced integration scheme for numerical integration. The full plate is discretized into 4x4 mesh. Averill and Reddy [6] compared the effect of full integration and selective reduced integration only on transverse deflection of plates for various shear deformable plate bending elements and recommended selective reduced integration scheme for both thick and thin plates. Rock and Hinton [7] reported 3x3 Gaussian quadrature as the appropriate rule of integration for both categories of plates modeled with 8-noded element.

The objective of this paper is to present a comparison of the performance of some simple plate bending finite elements, when applied to static and free vibration analyses of laminated composites. 4-noded, 8-noded and 9-noded elements based on first-order shear deformation theory and 4-noded element based on higher-order shear deformation theory are considered.

Finite Element Formulation

Four types of elements, namely, 4-noded, 8-noded and 9-noded isoparametric elements based on First-order Shear Deformation Theory (FSDT) and a 4-noded element based on Higher-order Shear Deformation Theory (HSDT), are used to model the laminated plate. A laminated plate with its geometry and reference axes is shown in Fig.1.

The displacement field based on first-order shear deformation theory is given by

$$u(x,y,z) = u_o(x,y) - z\,\theta_x(x,y)$$
$$v(x,y,z) = v_o(x,y) - z\,\theta_y(x,y)$$
$$w(x,y,z) = w_o(x,y) \qquad (1)$$

where u_o, v_o, w_o, θ_x and θ_y are chosen as the nodal degrees of freedom.

The constitutive matrix of the laminate is evaluated by integrating the constitutive matrices of the laminae through the thickness [8]. Element stiffness matrix and consistent load vector are evaluated using the standard procedure [9].

First-order shear deformation theory assumes a constant distribution of shear strain across the thickness of the

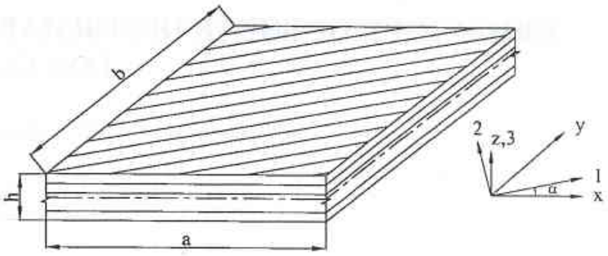

Fig. 1 Geometry of a laminated plate

plate. Elements based on this theory require a correction factor for the transverse shear stiffness terms. A higher-order shear deformation theory, which assumes a parabolic variation of transverse shear strains across the plate thickness, satisfies the condition of zero transverse shear stresses at top and bottom surfaces of the plate. Use of shear correction factor is not necessary in this case.

A 4-noded non-conforming element with seven degrees of freedom per node, namely, u_o, v_o, w_o, $\partial w_o / \partial x$, $\partial x, \partial w_o / \partial_y$, θ_x and θ_y is used to model the plate. The same shape functions that are used in the isoparametric formulation are used to interpolate u_o, v_o, θ_x and θ_y, but a non-conforming shape function based on Hermitian interpolation is used to interpolate the transverse displacement, w, as given by Ghosh and Dey[2]. The displacement field used in higher-order theory is of the form,

$$u(x,y,z) = u_o(x,y) - z\,\theta_x(x,y) - z^2\,\psi_x(x,y) - z^3\,\xi_x(x,y)$$

$$v(x,y,z) = v_o(x,y) - z\,\theta_y(x,y) - z^2\,\psi_y(x,y) - z^3\,\xi_y(x,y)$$

$$w(x,y,z) = w_o(x,y) \qquad (2)$$

Setting $\gamma_{xz}\left(x, y, \pm\frac{h}{2}\right) = \gamma_{yz}\left(x, y, \pm\frac{h}{2}\right) = 0$, Eq. (2) reduces to

$$u = u_o - z\left[\theta_x + \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\theta_x - \frac{\partial w_o}{\partial x}\right)\right]$$

$$v = v_o - z\left[\theta_y + \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\theta_y - \frac{\partial w_o}{\partial y}\right)\right]$$

$$w = w_o \qquad (3)$$

The detailed formulation of element stiffness matrix and consistent load vector is given by Ghosh and Dey [2].

Static Analysis

Effect of Order of Integration

First-order Shear Deformation Theory

Gauss quadrature has been used to evaluate the integrals in the expressions for stiffness matrix and nodal load vector.

Numerical results are obtained to study the effect of order of integration on the accuracy of results in the case of all the elements under consideration. A square symmetric 4-layer cross-ply (0/90/90/0) laminate, simply supported at all edges and subjected to a sinusoidal loading, is analysed. Two different thicknesses (b/h = 10 and b/h = 100) are considered. Results are presented in non-dimensional form as given below for the assumed values of $E_1/E_2 = 25$, $G_{12}/E_2 = G_{13}/E_2 = 0.5$, $G_{23}/E_2 = 0.2$ and $\nu_{12} = 0.25$.

$$\bar{w} = 100\left(\frac{w\,h^3 E_2}{q\,b^4}\right), \quad w = w\left(\frac{a}{2}, \frac{b}{2}\right)$$

$$\bar{\sigma}_x = \sigma_x\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{2}\right)\frac{h^2}{q\,b^2}, \quad \bar{\sigma}_y = \sigma_y\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{4}\right)\frac{h^2}{q\,b^2}$$

$$\bar{\tau}_{xy} = \tau_{xy}\left(0, 0, \frac{h}{2}\right)\frac{h^2}{q\,b^2}, \quad \bar{\tau}_{xz} = \tau_{xz}\left(0, \frac{b}{2}, 0\right)\frac{h}{q\,b}$$

$$\bar{\tau}_{yz} = \tau_{yz}\left(\frac{a}{2}, 0, 0\right)\frac{h}{q\,b}$$

The non-dimensional central deflection and stresses obtained based on full integration (i.e., 2x2 integration for both bending and shear terms in the case of 4-noded element and 3x3 integration for both bending and shear terms in the case of 8-noded and 9-noded elements), selective reduced integration (i.e., 2x2 integration for bending terms and 1x1 integration for shear terms in the case of 4-noded element and 3x3 integration for bending terms and 2x2 integration for shear terms in the case of 8-noded and 9-noded elements) and reduced integration (i.e., 1x1 integration for both bending and shear terms in the case of 4-noded element and 2x2 integration for both bending and shear terms in the case of 8-noded and 9-noded elements) are given in Table-1. The results are presented for a mesh division of 8x8 for the full plate.

In the case of 4-noded element, it is seen that the full integration scheme gives reasonable results for thick plates, but the results predicted for thin plates are erroneous. It is also seen that the selective reduced integration and reduced integration schemes give fairly good results for both thin and thick plates, with the reduced integration scheme results being nearer to the elasticity solution. Hence, further studies on 4-noded elements are carried out using reduced integration scheme.

In the case of 8-noded element, it is observed that the order of integration has no significant effect in the case of thick plates (b/h = 10). But in the case of thin plates, selective reduced integration and reduced integration schemes give better results than full integration. Results obtained by using selective reduced integration and reduced integration are practically the same. Hence, further studies on 8-noded elements are carried out using reduced integration, i.e., 2x2 integration for both bending and shear terms.

The behaviour of 9-noded element follows the same trend as that of 8-noded element. It is also seen that 8-noded and 9-noded elements give practically the same results. Hence, in the case of 9-noded element also, reduced integration scheme is sufficient so as to minimize the computational effort. It is reported in literature [1,3,6] that a selective reduced integration scheme is used for the analysis. The present study indicates that this unnecessarily increases the cost of evaluation for the same accuracy.

Higher-order Shear Deformation Theory

The same example used in first-order shear deformation theory is considered to study the effect of order of integration on the accuracy of the results. Three different possible schemes of integration, viz., 3x3 Gauss integration for both bending and shear terms (full), 3x3 integration for bending terms and 2x2 integration for shear terms (selective reduced) and 2x2 integration for both bending and shear terms (reduced) have been tried and the non-dimensionalised results are presented in Table-2.

The results indicate that a reduced integration of order 2x2 gives reasonably good results both for thin and thick plates. But the results reported by Ghosh and Dey [2] are based on 3x3 integration both for thin and thick plates, which has been found unnecessary from the present study.

Convergence Study

To study the convergence characteristics of the finite elements considered, the same example used to study the effect of order of integration has been considered for two different thicknesses (b/h = 10 and b/h = 100). The results presented in Tables-3 and 4 are based on the respective order of integration decided as a result of the above study.

Tables-3 and 4 reveal that 16x16 mesh gives reasonably good results for all the four types of elements and hence subsequent results are obtained using this mesh division. It is also seen that 8-noded and 9-noded elements give practically the same results. A finer mesh is required in the case of 4-noded element based on FSDT to get the same degree of accuracy. But, the results have been reported to be converged even with 4x4 mesh for full plate [1,3,5]. The present study indicates that even though the deflection values converge for 8x8 mesh, the stresses converge only with 16x16 mesh.

It is seen from Table-4 that the stresses and deflections predicted by the formulation based on HSDT are very near to the elasticity solution. Transverse shear stresses are also predicted by HSDT with reasonable accuracy and satisfy the condition of zero values at top and bottom surfaces of the plate following a parabolic distribution across the thickness of the plate. This is not possible with elements based on first-order theory.

Transverse shear stresses play a major role in the studies related to delamination, strength characteristics

Table-1 : Non-dimensional Central Deflection and Stresses of a Square Simply Supported 4-Layer Cross-ply (0/90/90/0) Laminate (FSDT)

b/h	Order of Integration	Type of Element	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
10	Full	4-noded	0.6427	0.4691	0.3410	0.0227	0.1296	0.1078
		8-noded	0.6626	0.5052	0.3658	0.0244	0.1423	0.1104
		9-noded	0.6627	0.5050	0.3657	0.0244	0.1423	0.1104
	Sele. reduced	4-noded	0.6632	0.4852	0.3554	0.0236	0.1333	0.1043
		8-noded	0.6626	0.5060	0.3662	0.0244	0.1424	0.1104
		9-noded	0.6627	0.5058	0.3661	0.0244	0.1423	0.1104
	Reduced	4-noded	0.6686	0.4917	0.3588	0.0239	0.1337	0.1041
		8-noded	0.6626	0.5053	0.3661	0.0244	0.1424	0.1104
		9-noded	0.6627	0.5052	0.3661	0.0244	0.1424	0.1104
	Reddy [10]		0.6628	0.4989	0.3615	0.0241	0.1667	0.1292
	Elasticity soln. [11]		0.7435	0.5590	0.4010	0.0275	0.3010	0.1960
100	Full	4-noded	0.1034	0.1250	0.0628	0.4951	0.0992	0.1838
		8-noded	0.4317	0.5387	0.2680	0.0216	0.1521	0.0868
		9-noded	0.4318	0.5387	0.2680	0.0216	0.1519	0.0864
	Sele. reduced	4-noded	0.4284	0.5248	0.2637	0.0208	0.1427	0.0808
		8-noded	0.4336	0.5455	0.2740	0.0216	0.1519	0.0864
		9-noded	0.4337	0.5453	0.2739	0.0216	0.1517	0.0860
	Reduced	4-noded	0.4338	0.5313	0.2670	0.0210	0.1427	0.0808
		8-noded	0.4336	0.5452	0.2740	0.0216	0.1522	0.0866
		9-noded	0.4337	0.5450	0.2739	0.0216	0.1520	0.0862
	Reddy [10]		0.4337	0.5382	0.2705	0.0213	0.1780	0.1009
	Elasticity soln.[11]		0.4385	0.5390	0.2710	0.0214	0.3390	0.1390

and fracture mechanics. Under such circumstances, it is necessary to use the formulation based on HSDT to predict the response of the plate.

Free Vibration Analysis

The same elements used for static analysis are considered to study the response of laminated composite plates to free vibration. Once the displacement field and the stress-strain relations are established, the equation governing free vibration may be expressed as

$$([K] - \omega^2 [M])\{\varphi\} = 0 \tag{4}$$

where $[K]$ and $[M]$ are the global stiffness and mass matrices respectively, obtained by the assembly of corresponding element matrices, ω the frequency of vibration of the system in rad/sec and $\{\varphi\}$, the mode shape vector. Element mass matrix is developed based on HRZ lumping scheme [5,9] including both normal and rotary inertia. Subspace iteration technique [12] is used for the extraction of eigen values and eigen vectors.

Effect of Order of Integration

First-order Shear Deformation Theory

The stiffness matrices for various elements considered are evaluated based on the respective order of integration decided from the study conducted for static analysis. The effect of order of integration for mass matrix on the accuracy of results has been studied by considering the same example used in the static analysis with the following properties. $E_1/E_2 = 40$, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = 0.25$, $\rho = 1$. The results presented for a mesh division of 8x8 for full plate are non-dimensionalised as

$$\bar{\omega} = 10\,\omega \sqrt{\frac{\rho h^2}{E_2}}\ .$$

The non-dimensional fundamental frequency of vibration based on different orders of integration for mass terms and using FSDT have been obtained and the results are presented in Table-5. For 4-noded element, 2x2 integration and 1x1 integration for mass terms have been tried, whereas, 3x3 integration and 2x2 integration have been tried for 8-noded and 9-noded elements.

Table-5 reveals that both reduced and exact integration for mass terms give exactly the same results in the case of 4-noded element. Hence, reduced integration is adopted for further studies. In the case of 8-noded and 9-noded elements also reduced integration is found to be sufficient.

Higher-order Shear Deformation Theory

The evaluation of element mass matrix is done as given by Ghosh and Dey [4]. The results obtained for the same example are presented in Table-6.

From this table, it is seen that 3x3 integration for mass terms gives results nearer to elasticity solution. Hence, 3x3 integration is adopted for further analysis. Tables-5 and 6 indicate that there is no significant difference between the predictions of first-order theory and higher-order theory. However, higher-order theory does not require shear correction factors.

Table-2 : Non-dimensional Central Deflection and Stresses of a Square Simply Supported 4-Layer Cross-ply (0/90/90/0) Laminate (4-noded Element, HSDT)

b/h	Order of Integration	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
10	Full	0.7166	0.5607	0.3910	0.0264	0.2522	0.1093
	Sele. reduced	0.7174	0.5621	0.3937	0.0260	0.2708	0.1560
	Reduced	0.7174	0.5621	0.3937	0.0260	0.2708	0.1560
	Reddy [10]	0.7147	0.5456	0.3888	0.0268	0.2640	0.1531
	Elasticity soln.[11]	0.7435	0.5590	0.4010	0.0275	0.3010	0.1960
100	Full	0.4294	0.5341	0.2675	0.0212	Unstable	Unstable
	Sele. reduced	0.4340	0.5502	0.2702	0.0212	0.2511	0.0962
	Reduced	0.4340	0.5502	0.2702	0.0212	0.2511	0.0962
	Reddy [10]	0.4343	0.5387	0.2708	0.0213	0.2897	0.1117
	Elasticity soln. [11]	0.4385	0.5390	0.2710	0.0214	0.3390	0.1390

Conclusions

The study indicates that the element stiffness matrix evaluated based on reduced integration scheme gives sufficiently accurate results both for thin and thick plates, irrespective of the finite element used. It has also been concluded that a 8x8 uniform mesh for quarter plate is required to predict the response accurately. The element based on higher-order theory predicts transverse shear stresses accurately which is not possible in the first-order shear deformation theory. It has also been observed that the values of transverse displacements and normal stresses are improved in higher-order shear deformation theory. But, in the case of free vibration analysis, the order of integration for mass terms does not have much influence on the frequency of vibration. Moreover, both first-order shear deformation theory and higher-order shear deforma-

Table-3 : Convergence Study of Central Deflection and Stresses of a Square Simply Supported 4-Layer Cross-ply (0/90/90/0) Laminate (FSDT)

b/h	Type of Element	Mesh Size	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
10	4-noded	4 x 4	0.6880	0.4705	0.3515	0.0232	0.1180	0.0940
		8 x 8	0.6686	0.4917	0.3588	0.0239	0.1334	0.1041
		16 x 16	0.6641	0.4971	0.3608	0.0241	0.1374	0.1067
		32 x 32	0.6631	0.4984	0.3613	0.0241	0.1385	0.1074
	8-noded	4 x 4	0.6615	0.5254	0.3807	0.0254	0.1525	0.1185
		8 x 8	0.6626	0.5053	0.3661	0.0244	0.1424	0.1104
		16 x 16	0.6627	0.5005	0.3626	0.0242	0.1397	0.1083
		32 x 32	0.6635	0.4997	0.3619	0.0242	0.1391	0.1078
	9-noded	4 x 4	0.6633	0.5227	0.3788	0.0252	0.1523	0.1181
		8 x 8	0.6627	0.5051	0.3660	0.0244	0.1424	0.1104
		16 x 16	0.6627	0.5005	0.3626	0.0242	0.1397	0.1083
		32 x 32	0.6635	0.4997	0.3619	0.0242	0.1391	0.1078
	Reddy [10]		0.6628	0.4989	0.3615	0.0241	0.1667	0.1292
	Elasticity soln. [11]		0.7435	0.5590	0.4010	0.0275	0.3010	0.1960
100	4-noded	4 x 4	0.4342	0.5113	0.2571	0.0203	0.1268	0.0719
		8 x 8	0.4338	0.5313	0.2670	0.0210	0.1427	0.0808
		16 x 16	0.4337	0.5365	0.2696	0.0212	0.1469	0.0832
		32 x 32	0.4337	0.5378	0.2702	0.0212	0.1479	0.0838
	8-noded	4 x 4	0.4319	0.5658	0.2845	0.0224	0.1736	0.1201
		8 x 8	0.4336	0.5452	0.2740	0.0216	0.1522	0.0866
		16 x 16	0.4337	0.5400	0.2713	0.0214	0.1492	0.0846
		32 x 32	0.4343	0.5391	0.2709	0.0213	0.1486	0.0842
	9-noded	4 x 4	0.4339	0.5640	0.2834	0.0223	0.1627	0.0921
		8 x 8	0.4337	0.5450	0.2739	0.0216	0.1520	0.0862
		16 x 16	0.4337	0.5399	0.2713	0.0214	0.1492	0.0846
		32 x 32	0.4343	0.5391	0.2709	0.0213	0.1486	0.0842
	Reddy [10]		0.4337	0.5382	0.2705	0.0213	0.1780	0.1009
	Elasticity soln.[11]		0.4385	0.5390	0.2710	0.0214	0.3390	0.1390

tion theory give very good results for free vibration analysis and the difference between them is not significant.

In short, 4-noded element based on higher-order shear deformation theory is recommended for static analysis. 16x16 mesh division for full plate is necessary for getting acceptable convergence. Element based on first-order shear deformation theory is sufficient for free vibration analysis only.

Table-4 : Convergence Study of Central Deflection and Stresses of a Square Simply Supported 4-Layer Cross-ply (0/90/90/0) Laminate (4-noded Element, HSDT)

b/h	Mesh Size	$\overline{w}$	$\overline{\sigma}_x$	$\overline{\sigma}_y$	$\overline{\tau}_{xy}$	$\overline{\tau}_{xz}$	$\overline{\tau}_{yz}$
10	4 x 4	0.7255	0.6094	0.3940	0.0255	0.2837	0.1148
	8 x 8	0.7174	0.5621	0.3937	0.0260	0.2708	0.1560
	16 x 16	0.7153	0.5497	0.3902	0.0266	0.2658	0.1542
	32 x 32	0.7149	0.5466	0.3892	0.0267	0.2644	0.1534
	64 x 64	0.7148	0.5458	0.3889	0.0267	0.2641	0.1531
	FSDT*	0.6641	0.4971	0.3608	0.0241	0.1374	0.1067
	Reddy [10]	0.7147	0.5456	0.3888	0.0268	0.2640	0.1531
	Elasticity soln. [11]	0.7435	0.5590	0.4010	0.0275	0.3010	0.1960
100	4 x 4	0.4076	0.5053	0.2515	0.0206	Unstable	Unstable
	8 x 8	0.4340	0.5502	0.2702	0.0212	0.2511	0.0962
	16 x 16	0.4346	0.5431	0.2714	0.0212	0.2890	0.1066
	32 x 32	0.4344	0.5398	0.2711	0.0213	0.2900	0.1113
	64 x 64	0.4343	0.5390	0.2709	0.0213	0.2898	0.1117
	FSDT*	0.4337	0.5365	0.2696	0.0212	0.1469	0.0832
	Reddy [10]	0.4343	0.5387	0.2708	0.0213	0.2897	0.1117
	Elasticity soln. [11]	0.4385	0.5390	0.2710	0.0214	0.3390	0.1390

* Results from the present study, (4-noded element, FSDT)

Table-5 : Non-dimensional Fundamental Frequency of a Square Simply Supported 4-Layer Cross-ply (0/90/90/0) Laminate (FSDT)

Type of Element	Order of Integration		$\overline{\omega}$	
	Stiffness	Mass	b/h = 5	b/h = 100
4-noded	1 x 1	1 x 1	4.2523	0.0186
		2 x 2	4.2523	0.0186
8-noded	2 x 2	2 x 2	4.3369	0.0188
		3 x 3	4.3349	0.0188
9-noded	2 x 2	2 x 2	4.3447	0.0188
		3 x 3	4.3447	0.0188
	Khdeir [13]		4.3416	-
	Elasticity soln. [14]		4.3006	-

Table-6 : Non-dimensional Fundamental Frequency of a Square Simply Supported 4-Layer Cross-ply (0/90/90/0) Laminate (4-noded Element, HSDT)

Order of Integration		$\overline{\omega}$	
Stiffness	Mass	b/h = 5	b/h = 100
2 x 2	2 x 2	4.3135	0.0188
	3 x 3	4.3061	0.0188
FSDT*		4.2523	0.0186
Khdeir [13]		4.3148	-
Elasticity soln. [14]		4.3006	-

* Results from the present study, (4-noded element, FSDT)

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