

DYNAMIC STABILITY OF SIMPLY SUPPORTED TAPERED BEAM WITH THERMAL GRADIENT BY FINITE ELEMENT METHOD

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Abstract

Effect of thermal gradient on the natural frequency, buckling load and dynamic stability of a simply supported tapered beam under a pulsating axial load is investigated by finite element method. A linear variation of the Young's modulus of the beam material, due to a steady one-dimensional thermal gradient is assumed. It is observed that the natural frequency and the buckling load of the beam decrease with increase in thermal gradient and thermal gradient has a destabilizing effect on the beam.

Notation

$A(x)$	= area of cross-section of the tapered beam at any section
E_1	= modulus of elasticity at section $x=0$
$E(x)$	= modulus of elasticity at any section x
$I(x)$	= moment of inertia at any section x
I_1	= moment of inertia of the section at $x = 0$
$[K_e]$	= assemblage stiffness matrix
$[K_e]_e$	= element elastic stiffness matrix
l	= length of the beam element
$[M]$	= assemblage mass matrix
$[M]_e$	= element mass matrix
P	= axial periodic load
P^*	= fundamental buckling load
P_o, P_t	= time independent amplitudes of loads
$\{q\}$	= assemblage nodal displacement vector
$\{\dot{q}\} = \frac{\partial}{\partial t} \{q\}$	
$\{q\}_e$	= element nodal displacement vector
$[S]$	= assemblage stability matrix
$[S]_e$	= element stability matrix
t	= variable time
T	= kinetic energy
U	= potential energy
v	= transverse displacement of the node
x	= coordinate along the length of the beam
α	= static load factor
β	= dynamic load factor
δ	= thermal gradient factor
ρ	= mass density of the material of the beam
Ω	= disturbing frequency
ω_1	= fundamental natural frequency without thermal gradient

$$\lambda_b = \text{buckling load parameter} \left(= \frac{P^* \times L^2}{E_1 I_1} \right)$$

$$\lambda_\omega = \text{natural frequency parameter}$$

$$\left(= \frac{\omega_1}{\sqrt{E_1 I_1 / (\rho \times A_1 \times L^4)}} \right)$$

$$\lambda_\Omega = \text{disturbing axial frequency parameter} \left(= \frac{\Omega}{\omega_1} \right)$$

Introduction

The parametric instability of structural elements is of major concern to mechanical and structural engineers. Structural members with a thermal gradient along its length and subjected to axial periodic forces varying with time are frequently encountered. These forces may result in parametric vibrations which can damage the structural element because of large amplitude of oscillations.

The stability of lateral motion of a uniform bar subjected to pulsating periodic axial loads was first studied by Baliaev [1] and latter by Mettler [2] and others [3,4,5] and is well documented in the book by Bolotin [6]. Brown et.al [7] studied the dynamic stability of uniform bars with various boundary conditions using finite element method. Ahuja and Duffield [8] investigated the same problem using a slightly modified Galarkin method. A discrete element type of numerical approach was employed by Burney and Jaeger [9] to study the parametric instability of a uniform column. Iwatsubo et.al [10,11] investigated the existence of different types of resonances for clamped and clamped - simply supported columns under periodic axial loads by finite difference method and the existence

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of different types of resonances in columns for four different boundary conditions analytically. Datta and Chakrabarty [12] investigated the stability of a tapered beam by finite element method. Abbas [13,14] studied free vibration of Timoshenko beam with elasticity supported ends using finite element model. He also studied the effects of rotational speed and root flexibility on the static buckling loads and on the regions of dynamic instability of a Timoshenko beam by the same method. Yokoyama [15] in a recent work investigated the effect of an elastic foundation support on the static buckling load, natural frequency and stability of a uniform Timoshenko beam by finite element method. Lien and Der [16] studied the stability behaviour of a Timoshenko beam subjected to a uniformly distributed follower force by finite element method. Datta and Lal [17] studied the static buckling characteristics of a non-prismatic bar with localized zones of damage and subjected to an intermediate axial load by finite element analysis.

The modulus of elasticity of the material is greatly affected by the temperature. In high-speed atmospheric flights and nuclear engineering application the mechanical and structural parts are subjected to high temperature and also experience fluctuating loads. For most of the engineering materials the Young's modulus varies linearly with temperature. Tomar and Jain studied the effect of thermal gradient on the frequencies of rotating beams with and without thermal gradient [18,19].

Effect of change in mechanical properties of the material, namely the Young's modulus due to thermal gradient, on the dynamic stability of the beam is to be studied for effective design of these structural components. The present work deals with the simple resonance of a parametrically excited simply supported tapered beam subjected to thermal gradient along its length. Finite element method is employed to carry out the analysis. The regions of instability were determined by Floquet theory.

Formulation of the Problem

Let the simply supported tapered beam shown in Fig. 1 is represented by an assembly of finite elements connected together at the nodes. A typical finite element is shown in Fig.2 with v_i, θ_i, v_j and θ_j as nodal displacements. The matrix equation for free vibration of axially loaded discretised system is

$$[M] \{\ddot{q}\} + [K_e] \{q\} - P [S] \{q\} = 0 \tag{1}$$

where $\{q\}$ = Assemblage nodal displacement vector $[v_i, \theta_i, v_j, \theta_j]^T$.

The dynamic load $P(t)$ is periodic and can be expressed in the form $P = P_0 + P_t \cos \Omega t$, where Ω is the disturbing frequency, P_0 the static and P_t the amplitude of time dependent component of the load, can be represented as the fraction of the fundamental static buckling load P^* . Hence substituting $P = \alpha P^* + \beta P^* \cos \Omega t$ with α and β as static and dynamic load factors respectively.

The equation (1) becomes

$$[M] \{\ddot{q}\} + \left([K_e] - \alpha P^* [S_s] - \beta P^* \cos \Omega t [S_t] \right) \{q\} = 0 \tag{2}$$

where the matrices $[S_s]$ and $[S_t]$ reflect the influence of P_0 and P_t respectively. If the static and time dependent component of loads are applied in the same manner, then $[S_s] = [S_t] = [S]$.

Equation (2) represents a system of second order differential equations with periodic coefficients of the Mathieu-Hill type. The development of region of instability arises from Floquet's theory which establishes the existence of the periodic solutions of period T and $2T$, where $T = \frac{2\pi}{\Omega}$. The boundaries of the primary instability region with period $2T$ are of practical importance

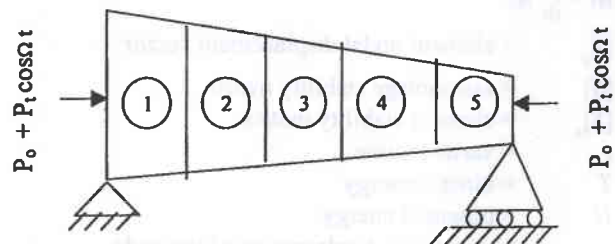


Fig. 1

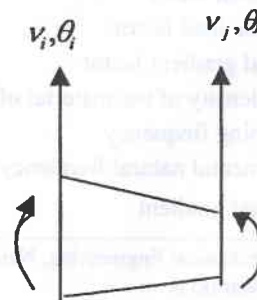


Fig. 2

(Bolotin - 5) and the solution can be achieved in the form of trigonometric series.

$$q(t) = \sum_{k=1}^{\infty} \left[\{a_k\} \sin \frac{K\theta t}{2} + \{b_k\} \cos \frac{K\theta t}{2} \right] \quad (3)$$

Putting this in equation (2) and if only first term of the series is considered, equating coefficients of $\sin \frac{\theta t}{2}$ and $\cos \frac{\theta t}{2}$ the equation becomes

$$\left[[K_e] - (\alpha \pm \beta/2)P^* [S] - \frac{\Omega^2}{4} [M] \right] \{q\} = 0 \quad (4)$$

Equation (4) represents an eigenvalue problem for unknown values of α , β and P^* . This equation gives two sets of eigenvalues (Ω) bounding the regions of instability due to presence of plus and minus sign.

Also this equation (4) represents the solution to a number of related problems.

- i) For free vibration : $\alpha = 0, \beta = 0$ and $\lambda = \frac{\Omega}{2}$

$$\text{Equation (4) becomes } \left([K_e] - \lambda^2 [M] \right) \{q\} = 0 \quad (5)$$

- ii) For vibration with static axial load:

$$\beta = 0, \alpha \neq 0, \lambda = \frac{\Omega}{2}$$

Equation (4) becomes

$$\left([K_e] - \alpha P^* [S] - \lambda^2 [M] \right) \{q\} = 0 \quad (6)$$

- iii) For static stability: $\alpha = 1, \beta = 0, \lambda = \frac{\Omega}{2}$

$$\text{Equation (3.4) becomes } \left([K_e] - P^* [S] \right) \{q\} = 0 \quad (7)$$

- iv) For dynamic stability, when all terms are present

$$\text{Let } \Omega = \left(\frac{\Omega}{\omega_1} \right) \omega_1$$

where ω_1 is the fundamental natural frequency as obtained from the solution of equation (5). Equation (4) then becomes

$$\left([K_e] - \left(\alpha \pm \frac{\beta}{2} \right) P^* [S] \right) \{q\} = \theta \frac{\omega_1^2}{4} [M] \{q\} \quad (8)$$

$$\text{where, } \theta = \left(\frac{\Omega}{\omega_1} \right)^2$$

The fundamental natural frequency ω_1 and critical static buckling load P^* can be solved using the equations (5) and (7) respectively. The regions of dynamic instability can be determined from equation (8).

Element Matrices

The increase in potential energy of an element length 'l' of a tapered beam subjected to an axial force 'P' is given by

$$U = \frac{1}{2} \int_0^l E(x) I(x) \left(\frac{d^2 v}{dx^2} \right)^2 dx - \frac{1}{2} P \int_0^l \left(\frac{dv}{dx} \right)^2 dx \quad (9)$$

Assuming polynomial expansion for v and substituting in equation (9) this equation becomes.

$$U = \frac{1}{2} \{q\}_e^T [K] \{q\}_e \quad (10)$$

$$\text{where } [K] = [K_e]_e - P [S]_e$$

The kinetic energy T for an elemental length l of a tapered beam is given by

$$T = \frac{1}{2} \int_0^l \rho A(x) \{v'\}^2 dx \quad (11)$$

where ρ is the mass density of the material of the beam. Using the polynomial expansion for v and substituting in equation (11) we get

$$T = \frac{1}{2} \{\dot{q}\}_e^T [M]_e \{\dot{q}\}_e \quad (12)$$

Element mass matrix, element elastic stiffness matrix and element stability matrix which is a function of the axial load P are given by the expressions

$$[M]_e = \int_0^L [N]^T \rho [N] dx \quad (13)$$

$$[K]_e = \int_0^L [N'']^T [D] [N''] dx \quad (14)$$

$$[S]_e = \int_0^L [N']^T [N'] dx \quad (15)$$

where $[N''] = \frac{\partial^2}{\partial x^2} [N]$, $[N'] = \frac{\partial}{\partial x} [N]$ and $[N]$ is the element shape function matrix

$$[D] = E(x) I(x).$$

The Young's modulus E is assumed to vary linearly along the length of beam due to thermal gradient.

$$E(x) = E_1 [1 - \delta (x/l)]$$

The overall matrices $[K]_e$, $[S]$ and $[M]$ are obtained by assembling the corresponding element matrices. The displacement vector consists of any active nodal displacements.

Results and Discussion

The following properties of the beam are taken for numerical computations :

Length of the beam = 1m; Cross-sectional dimension at the tip = 2 x 2 cm; Cross - sectional dimension at the root = 12 x 12 cm; Material mass density of the aluminum beam = 2800 kg/m³ $E_1 = 70 \times 10^9$ N/m²

Natural Frequency and Buckling Load

Five-element discretisation of the beam is used to evaluate the buckling load and natural frequency. In order to check the accuracy of discretisation, buckling load for fixed free end conditions and without thermal gradient is calculated from the present formulation and compared with the theoretical result of Timoshenko [20], which shows good agreement.

Present FEM	Timoshenko (20)
2.3988×10^2 KN	2.3973×10^2 KN

Figure 3 and 4 shows the variation of natural frequency parameter and buckling load parameter respectively of the first three modes with thermal gradient parameter δ . The values of both the frequency parameter and buckling load parameter decrease with increase in the value of δ . This decrease is negligible for fundamental natural frequency and buckling load. The rate of decrease is more for higher modes.

Stability Analysis

Five-element discretisation of the beam is used for dynamic stability study. This gives rapid convergence of the boundary frequency for the first five instabilities zones.

In order to study the stability of the beam, instability regions are obtained for different values of thermal gradient factor δ and static factor α . These are shown in Figs. 5-14.

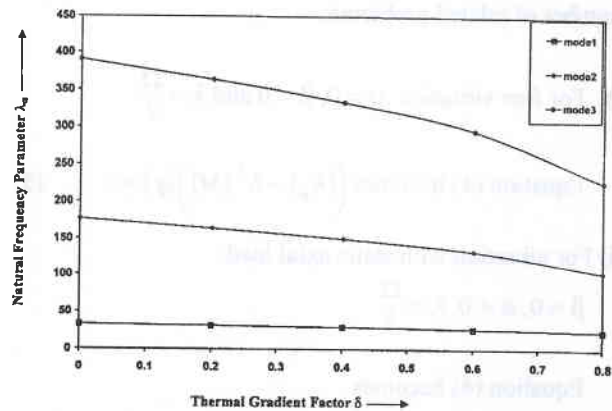


Fig. 3 Variation of natural frequency with thermal gradient

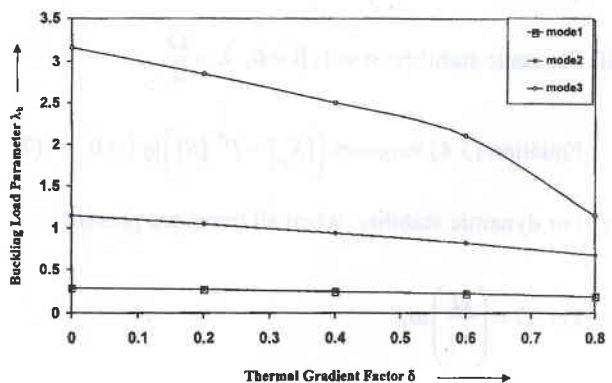


Fig. 4 Variation of buckling load with thermal gradient

Figures 5-7 shows the instability regions for $\alpha = 0.4$ and $\delta = 0.3$ and 0.6 respectively. It is seen that with thermal gradient present, the lower boundary of the first instability region curls towards the dynamic load factor axis, thereby making the first instability region wider. For higher values of thermal gradient factor the lower boundary truncates on

the β - axis at values of β less than one. For example, for $\alpha = 0.4$ and $\delta = 0.6$ the lower boundary of the first instability region truncates on the β - axis at $\beta = 0.8$. Truncation of lower boundary on the β - axis at values of β less than one indicates instability for amplitude of time dependent component of the load less than the fundamen-

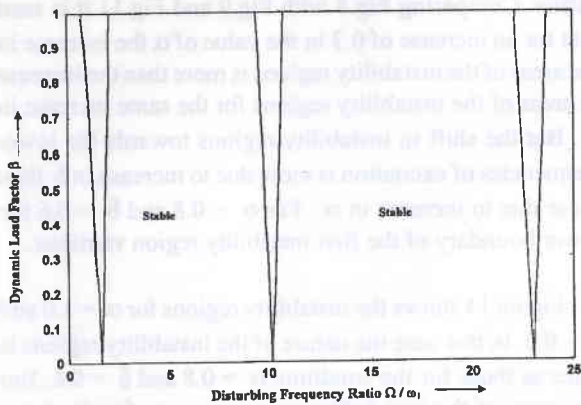


Fig. 5 Stability regions for $\alpha = 0.4 \delta = 0.0$

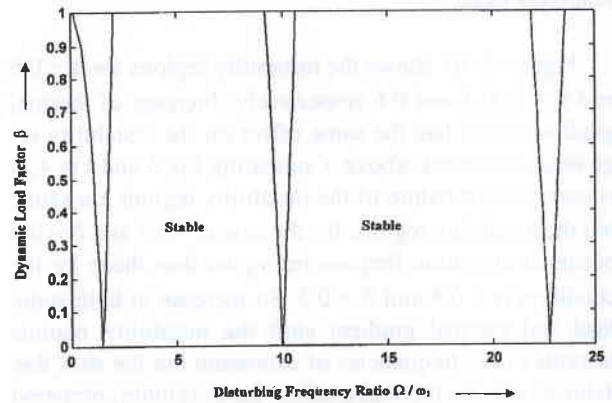


Fig. 8 Stability regions for $\alpha = 0.5 \delta = 0.0$

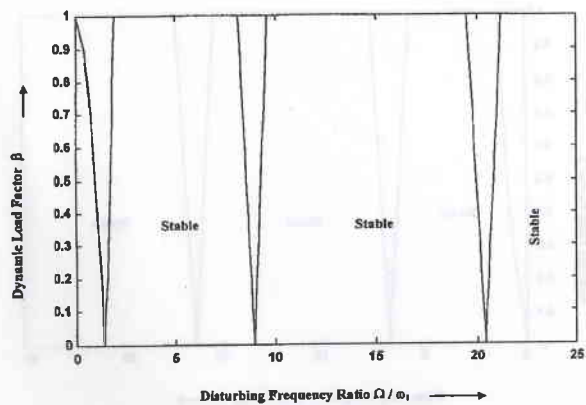


Fig. 6 Stability regions for $\alpha = 0.4 \delta = 0.3$

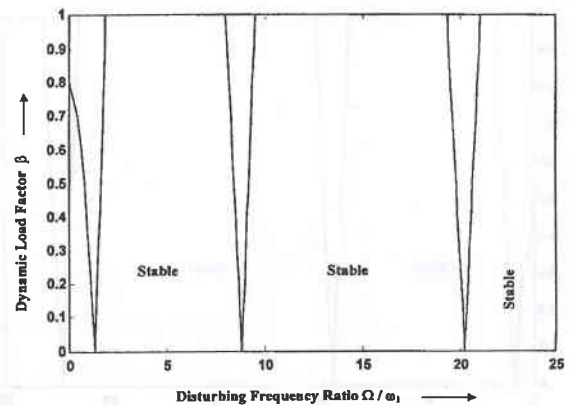


Fig. 9 Stability regions for $\alpha = 0.5, \delta = 0.3$

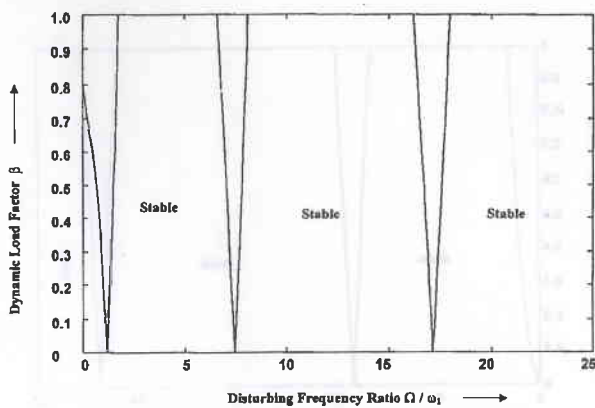


Fig. 7 Stability regions for $\alpha = 0.4 \delta = 0.6$

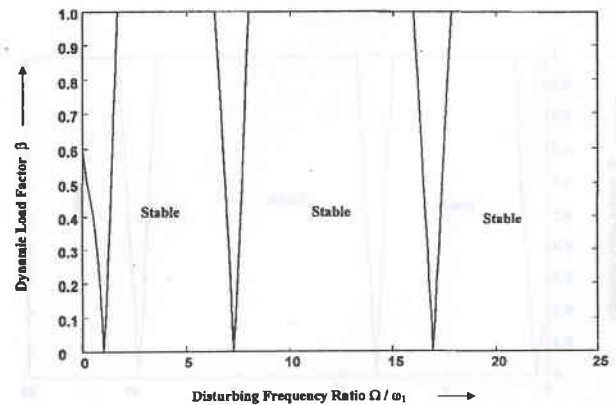


Fig. 10 Stability regions for $\alpha = 0.5, \delta = 0.6$

tal static buckling load. There is also increase in areas of the other two instabilities regions. Moreover increase in thermal gradient factor β shifts the instability regions towards the dynamic load factor axis, that is the instability occurs at lower frequency of excitation. This shift is less for the instability region but for other two regions it is relatively large.

Figures 8-10 shows the instability regions for $\alpha = 0.5$ and $\delta = 0, 0.3$ and 0.6 respectively. Increase of thermal gradient factor has the same effect on the instability regions as discussed above. Comparing Fig.6 and Fig.4, it is seen that the nature of the instability regions are same, but the instability regions for the case $\alpha = 0.5$ and $\delta = 0.0$ occurs at excitation frequencies higher than those for the condition $\alpha = 0.4$ and $\delta = 0.3$. So increase in both static load and thermal gradient shift the instability regions towards lower frequencies of excitation but the shift due to increase in the thermal gradient factor is more compared to that due to increase in static load factor for the same

point of truncation of the lower boundary of the first instability region.

Figures 11-13 shows the instability regions for $\alpha = 0.8$ and $\delta = 0, 0.3$ and 0.6 respectively. The instability regions show the same behaviour with increase in δ as discussed earlier. Comparing Fig.8 with Fig.9 and Fig.11 it is seen that for an increase of 0.3 in the value of α the increase in the areas of the instability regions is more than the increase in areas of the instability regions for the same increase in δ . But the shift in instability regions towards the lower frequencies of excitation is more due to increase in δ than those due to increase in α . For $\alpha = 0.8$ and $\delta = 0.6$ the lower boundary of the first instability region vanishes.

Figure 14 shows the instability regions for $\alpha = 1.0$ and $\delta = 0.0$. In this case the nature of the instability regions is same as those for the condition $\alpha = 0.8$ and $\delta = 0.6$. But the areas of the instability zones are more for the latter case.

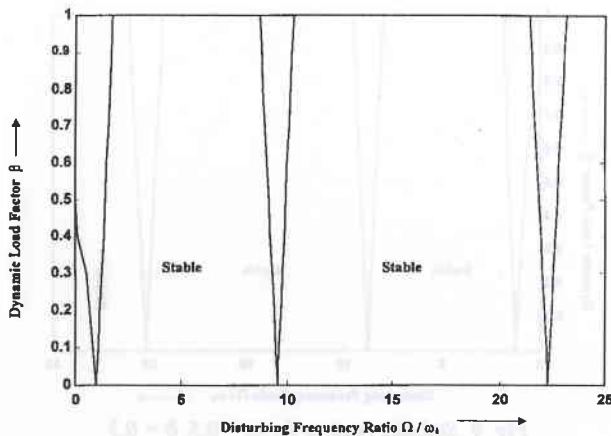


Fig. 11 Stability regions for $\alpha = 0.8, \delta = 0.0$

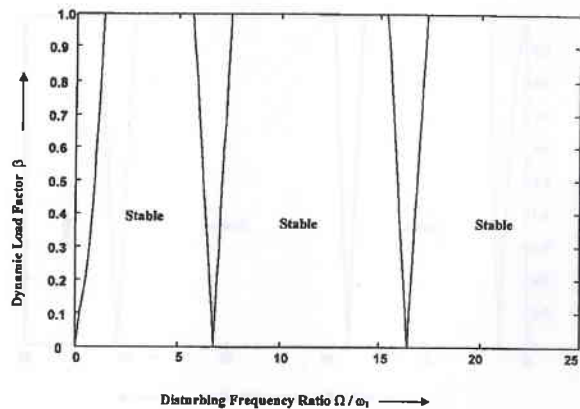


Fig. 13 Stability regions for $\alpha = 0.8, \delta = 0.6$

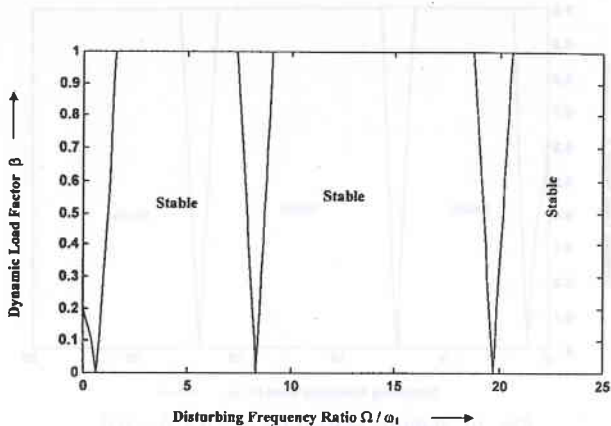


Fig. 12 Stability regions for $\alpha = 0.8, \delta = 0.3$

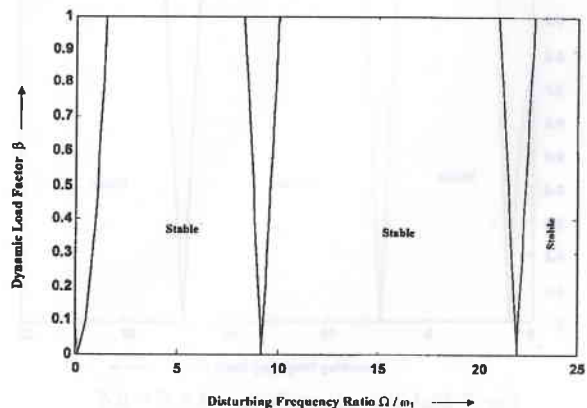


Fig. 14 Stability regions for $\alpha = 1.0, \delta = 0.0$

Conclusions

A finite element method is presented for the stability analysis of a tapered beam with thermal gradient subjected to axial periodic load. Increase in thermal gradient decreases the natural frequency and the static buckling load. There is an increase in the areas of instability regions and the instability region shift to lower frequencies of excitation with increase in thermal gradient. Increase in static load or thermal gradient, increases the areas of the instability regions, and shift the regions towards lower frequencies of excitations. Static load has a greater influence on increasing the areas whereas increase in thermal gradient has a greater influence in shifting the regions towards lower frequencies of excitations.

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