# VIBRATION ANALYSIS OF THIN CIRCULAR DISC INTERACTING WITH A FLUID IN CYLINDRICAL CONTAINER

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**Abstract**

*In this paper, vibration analysis of a thin circular disc interacting with a fluid in cylindrical container is carried out. It has been done systematically by carrying our mathematical modeling and subsequently validated by ANSYS software. The author has first determined natural frequencies and mode shapes for a thin circular disc alone and then for fluid contained by a cylindrical vessel alone. The author had then found natural frequencies and mode shapes for coupled case of thin circular disc interacting with a fluid contained by the cylindrical container. The same had been validated by ANSYS software. It is found that natural frequencies of coupled vibrations are different than that of the frequency of disc or fluid taken separately. In order to study the correct failure pattern of the coupled vibration system, it, thus, becomes a necessity to study coupled vibration analysis.*

## Nomenclature

a = radius of the circular disc
c = speed of sound
D = disc constant, ($D = Eh^3/12\,(1-\nu^2)$)
E = modulus of elasticity of plate material
f = natural frequency (Hertz)
h = thickness of disc
$I_0$ = principal inertia (mass per unit area), $I_0 = \rho h$
$I_2$ = rotary inertia, ($I_2 = \rho h^3/12$)
$I_n$ = modified Bessel function of first kind
$J_n$ = bessel function of first kind
$K_n$ = modified Bessel function of second kind
m = number of nodal circles for disc, number of nodal diameters in coupled solution
n = number of nodal diameters in case of analysis of disc only
p = fluid pressure
q = number of radial wave in the acoustic media
r = radius of the cylinder at a given point
t = time co-ordinate
$W_n$ = deflection of $n^{th}$ mode
$W(r,\theta)$ = a function of only r and $\theta$
$Y_n$ = bessel function of second kind
z = z-coordinate
$\omega$ = natural frequency (radians/sec)
$\rho_l$ = density of fluid
$\rho_d$ = density of disc material
$w_0$ = deflection in z-direction
$\phi$ = velocity potential

## Introduction

Vibration analysis of a machine or a structure is of great importance in mechanical design. Many harmful effects like excessive stresses, undesirable noise, looseness of part and partial or complete failure of parts, which occur due to vibrations, can be predicted by the vibrational analysis. The frequency range within which it can operate safely can also be determined and thus one may avoid damage to the machine/structure. Nearly all solid structures in real life exist in surface contact with one or more fluid media, of which the most common are the air and water. Vibration generated in a solid structure is communicated to a fluid with which it is in contact via normal motion of the media interface. Everyday examples include the generation of audible sound in the air by vibrating surfaces such as machines, building components and stringed instruments. The interaction between structural and fluid systems can alter the free and forced vibration behaviour of the coupled components from their uncoupled forms.

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Warburton [1] has carried out vibration analysis of rectangular plates backed by a rectangular cavity where the mode shapes, free and forced frequencies of the vibrating system were calculated. Bokil and Shirahatti [2] have studied the sound-structure interaction problems involving the fluid in a cavity having one flexible wall. In the paper, they have developed a technique for sound-structure interaction problems. It has been shown in the paper that how one can synthesize the modal properties of the interacting system from those of the non-interacting fluid and the structural systems. Scarpa [3] has studied the parametric sensitivity analysis of coupled acoustic-structural systems. Here a model for parametric sensitivity analysis of coupled acoustic structural systems is presented. Coupled frequencies and modes have been calculated using Eulerian formulation. Maity and Bhattacharyya [4] have done a parametric study on fluid-structure interaction problems. They have studied fluid-structure system considering the coupled effect of elastic structure and fluid using Finite Element Method. The equations of motion of the fluid, considering it as inviscid and compressible, are expressed in terms of the pressure variable alone. The solution of the coupled system is accomplished by solving the two systems separately with the interaction effects at the fluid-solid interface enforced by a developed iterative scheme. The parametric study of the coupled system shows the importance of fluid height and material property of the structure. Gorman and Ding [5] have studied the free vibrations of rectangular plates by using superposition-Galerkin method. Traditional superposition method, used for analyzing the plate vibrations, is very demanding and a difficult task. In this paper, modified superposition-Galerkin method had been introduced. The modified superposition-Galerkin method used have eliminated above mentioned problem and also given excellent results. Airey [6] has studied the vibrations of circular plates and their relation to Bessel functions having various boundary conditions. Frequencies for free vibration for circular disc clamped at its periphery have also been found by him. Amabili [7] has studied the free vibrations of a circular cylindrical tank partially filled with an inviscid and incompressible liquid with a free surface orthogonal to the tank axis. The tank is modeled by a simply supported cylindrical shell connected to a simply supported circular plate. The modes of vibration of the structure are investigated and the solution is obtained as an eigenvalues problem by using the Rayleigh-Ritz expansion of the mode shapes. Lee, Yeo and Samoilenko[8] have studied the effect of the number of nodal diameters on non-linear interactions in asymmetric vibrations of a circular plate.

In this paper, vibration analysis of a thin circular disc clamped at its periphery is first studied. The deflection of the disc is derived by assuming solution in terms of Fourier series. For this, a mathematical formulation for a thin circular disc, clamped at its periphery, is made and its natural frequencies and mode shapes are determined. This mathematical formulation is validated on ANSYS 10. A solid 3-D finite strain 190 element, from element library of ANSYS 10, is chosen for the analysis. Quantative comparison of the results of these two formulations for few nodal diameters is made and found to be in close agreement. Then, vibration analysis of fluid inside a closed cylindrical container is studied mathematically by assuming fluid as a compressible and non-viscous fluid and container walls as rigid one. It is validated by ANSYS 10 quantatively for few nodal circle and nodal diameter. For this purpose, FLUID 30 element is chosen for the fluid and SOLID 95 element for the container walls from the ANSYS element library. Finally, a coupled vibration analysis of circular disc interacting with a compressible and non-viscous fluid is then studied by making a mathematical modeling and the same is validated by ANSYS 10. This analysis is carried out by selecting SHELL 63 element for the disc, FLUID 30 element for the fluid and SOLID95 element for the container walls from the ANSYS element library. For the coupled vibration analysis, results obtained by ANSYS 10 and modeled are compared quantatively for few nodal diameter and nodal circle and found to be in close agreement to each other. It is further noticed by the author that frequencies for the circular disc interacting with the fluid contained inside the cylindrical cavity are different than those obtained for the circular disc and fluid alone. The difference in these two cases will be more appreciable and prominent if the length of the cylindrical cavity is increased.

## Free Vibrations of a Circular Disc

The equation of motion of a thin circular disc in polar coordinates (Fig.1) under the action of forces and neglecting the effect of pre-stress in the disc is given by [9]

$$D\,\nabla^2\nabla^2 w_0 + I_0\frac{\partial^2 w_0}{\partial t^2} - I_2\frac{\partial^2}{\partial t^2}(\nabla^2 w_0) = 0 \tag{1}$$

where

$$\nabla^2 = \nabla\,.\,\nabla = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\,\theta^2}\right)$$

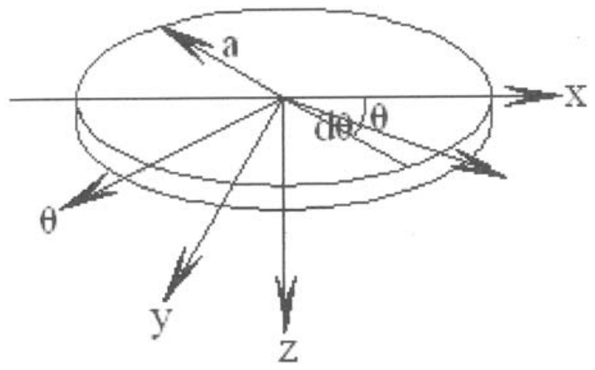


*Fig.1 Polar co-ordinate system for a disc*

In Eq. (1), D is disc constant (= $Eh^3/(1-\nu^2)$), $\nabla^2$ is the Laplace operator and defined above, $w_0$ is the deflection in z-direction, $I_0$ is the principal inertia (mass per unit area $I_0 = \rho h$) and $I_2$ is the rotary moment of inertia ($I_2 = \rho h^3/12$).

When the disc is assumed to have free vibration, the deflection is periodic and can be expressed as:

$$w_0 (r, \theta, t) = W(r, \theta) \cos (\omega t) \quad (2)$$

where ω is the circular frequency of vibration (rad/s) and W is a function of $r$ and θ only. Substituting Eq. (2) into Eq. (1), one gets

$$D \nabla^2 \nabla^2 W - I_0 \omega^2 W + I_2 \omega^2 \nabla^2 W = 0 \quad (3)$$

The presence of the rotary inertia, $I_2$ creates difficulties in solving the equation of motion while it contributes little to the frequencies, especially to the fundamental frequency. Hence, one may neglect the rotary inertia term safely. Thus, Eq. (3) reduces to

$$(\nabla^4 - \beta^4) W = 0$$

or

$$(\nabla^2 + \beta^2)(\nabla^2 - \beta^2) W = 0$$

or

$$\nabla^2 W_1 + \beta^2 W_1 = 0$$

$$\nabla^2 W_2 + \beta^2 W_2 = 0 \quad (4)$$

where,

$$\beta^4 = \frac{I_0 \omega^2}{D}$$

We assume solution of Eq. (4) in the form of the general Fourier series

$$W(r, \theta) = \sum_{n=0}^{\infty} W_n(r) \cos n\theta + \sum_{n=1}^{\infty} W_n^*(r) \sin n\theta \quad (5)$$

where, $W_n(r)$ and $W_n^*(r)$ are the functions of $r$ only giving the radial mode shape.

Substitution of Eq. (5) into Eq. (4) yield two equations in the form of Bessesl equations, these equations have the solution in the following form:

$$W_{n1} = A_n J_n (\beta r) + B_n Y_n (\beta r) \quad \text{and}$$

$$W_{n2} = C_n I_n (\beta r) + D_n K_n (\beta r) \quad (6)$$

where $A_n$, $B_n$, $C_n$, $D_n$ are the coefficients which are solved using the boundary conditions and determine the mode shapes, $J_n$ and $Y_n$ are the Bessel functions of first and second kind respectively, and $I_n$ and $K_n$ are the modified Bessel functions of the first and second kind respectively. Thus, The general solution of Eq. (4) is thus given [10] as:

$$W(r, \theta) = \sum_{n=0}^{\infty} \left[ A_n J_n (\beta r) + B_n Y_n (\beta r) + C_n I_n (\beta r) + D_n K_n (\beta r) \right] \cos n\theta$$

$$+ \sum_{n=0}^{\infty} \left[ A_n^* J_n (\beta r) + B_n^* Y_n (\beta r) + C_n^* I_n (\beta r) + D_n^* K_n (\beta r) \right] \sin n\theta \quad (7)$$

where $A_n^*, B_n^*, C_n^*, D_n^*$ are the coefficients which are solved using the boundary conditions. For solid circular plates, the terms involving $Y_n$ and $K_n$, must be discarded in order to avoid singularity (i.e., infinite values) of deflections and stresses at the origin, i.e. at $r = 0$. In addition to it, if the boundary conditions are symmetrically applied about a diameter of the disc then the second expression containing *sin* ($n\theta$) is not needed to represent the solution as it becomes symmetrical with *cos* ($n\theta$). Then the $n^{th}$ term of Eq. (7) becomes

$$W_n(r,\theta) = \left[A_n J_n(\beta r) + C_n I_n(\beta r)\right] \cos n\theta \tag{8}$$

where $n$ is nodal line and may vary as 0, 1, 2, ..., ∞. A *nodal line* is one which has zero deflection (i.e., $W_n = 0$). For circular plates, nodal lines are either concentric circles or diameters. The nodal diameters are determined by $n\theta = \pi/2,\ 3\pi/2, \ldots$ . The boundary conditions for the circular disc clamped at its periphery may be given as:

$$W_n = 0 \text{ and } \partial W_n/\partial r = 0 \text{ at } r = a \text{ for any } \theta \tag{9}$$

Using above boundary conditions in Eq. (8), one obtain

$$\begin{bmatrix} J_n(\lambda) & I_n(\lambda) \\ J'_n(\lambda) & I'_n(\lambda) \end{bmatrix} \begin{Bmatrix} A_n \\ C_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10}$$

where, $\lambda = \beta a$ and the prime denotes differentiation with respect to the argument, $\beta r$. For nontrivial solution, one can set the determinant of the coefficient matrix of Eq. (10) to zero, i.e.

$$\begin{vmatrix} J_n(\lambda) & I_n(\lambda) \\ J'_n(\lambda) & I'_n(\lambda) \end{vmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{11}$$

where, J and I are Bessel functions and modified Bessel functions of first kind respectively. Expanding the determinant and using the recursion relations given as

$$\lambda J'_n(\lambda) = n J_n(\lambda) - \lambda J_{n+1}(\lambda);$$

$$\lambda I'_n(\lambda) = n I_n(\lambda) + \lambda I_{n+1}(\lambda) \tag{12}$$

One can find, from Eq. (11), Frequency equation for the problem as

$$J_n(\lambda) I_{n+1}(\lambda) + I_n(\lambda) J_{n+1}(\lambda) = 0 \tag{13}$$

or

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)} + \frac{I_{n+1}(\lambda)}{I_n(\lambda)} = 0 \tag{14}$$

The roots, $\lambda$, of Eq. (14) are used to determine the frequencies($\omega$) of the circular disc as

$$\omega^2 = \frac{(D\beta^4)}{I_0} = \frac{(D\lambda^4)}{a^4 I_0} \tag{15}$$

The mode shape associated with $\lambda$ is determined using Eq. (11). From Eqs. (6) and (9), one gets

$$\frac{A_n}{C_n} = -\frac{I_n(\lambda)}{J_n(\lambda)} \tag{16}$$

where $\lambda$ is the solution (i.e., root) of Eq. (14). Therefore, the radial mode shape of vibration from Eq. (8) is given by the following relation:

$$W_n(r) = A_n J_n(\beta r) + C_n I_n(\beta r) \tag{17}$$

And using Eq. (16) in Eq. (17), and putting $C_n = 1$ one gets

$$W_n(r) = -\frac{I_n(\lambda)}{J_n(\lambda)} J_n(\beta r) + I_n(\beta r) \text{ where } \lambda = \beta a \tag{18}$$

Eq. (18) gives the modes of free vibration of a circular plate clamped at its periphery.

### Free Vibrations of Fluid inside the Cylindrical Container

The non-viscous and compressible fluid inside a cylindrical duct is schematically shown in the Fig.2. The 3-D wave equation in $r$, $\theta$ and $z$ coordinate systems is given [11] as

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{19}$$

where, $\phi$ is the velocity potential, $c$ is the speed of the sound, $r$ is the radius of cylinder at any plane and $t$ is time. The velocity potential is finite and is single valued inside the cylinder and vanishes on its boundary. These observations are expressed by the fact that velocity potential ($\phi$)

satisfies the following three pairs of boundary conditions for all time periods.

$\phi = 0$ at $z = 0$ and $\phi =$ at $z = L$;

$\phi =$ finite at $r = 0$ and $\phi = 0$ at $r = a$;

$$\phi_{(\phi = 0)} = \phi_{(\phi = 2\pi)} \text{ and } \phi'_{(\phi = 0)} = \phi'_{(\phi = 2\pi)} \tag{20}$$

The wave equation is solved by *separation of variables* method. It consists of finding those solutions which have the product form given as

$$\phi = R(r)\ \Theta(\theta)\ Z(z)\ T(t) \tag{21}$$

where $R(r)$ is a function of $r$ only, $\Theta(\theta)$ is a function of $\theta$ only, $Z(z)$ is a function of $z$ only and $T(t)$ is a function of $t$ only. Introducing it into the wave equation, Eq. (19), and dividing it by the product of these four factors, one gets

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}r\frac{dR}{dr} + \frac{1}{r^2}\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} - \frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = 0 \tag{22}$$

By bringing $z$-term to the right hand side, the resulting equality holds for all $r$, $\theta$, $z$ and $t$. Thus, the right hand must be independent of $r$, $\theta$ and $t$, while the left hand side must be independent of $z$. But the two sides are equal. Thus, the quantity (function) must be independent of $r$, $\theta$, $t$ and $z$, i.e., it must be some constant. Let it is $k^2$. So, $Z(z)$ satisfies the following

$$-\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2 \tag{23}$$

Next isolate the $\Theta$-term and by the analogous argument, one obtain

$$\frac{1}{r^2}\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = k^2 \text{ or } \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -m^2$$

$$\textit{where } -m^2 = r^2 k^2 \tag{24}$$

Its solution is of the form

$$\Theta(\theta) = C_1 e^{im\theta} + D_1 e^{-im\theta} = C_1(\cos m\theta + i\sin m\theta)$$

$$+ D_1(\cos m\theta - i\sin m\theta) \tag{25}$$

where $C_1$ and $D_1$ are constants. Similarly, one may obtain

$$\frac{1}{R}\left[\frac{1}{r}\frac{d}{dr}r\frac{dR}{dr} - \frac{m^2}{r^2}R\right] = -\lambda^2 \tag{26}$$

$$\text{or } r^2R'' + rR' + (\lambda^2 r^2 - m^2)\ R = 0 \tag{27}$$

Its solution is of the form

$$R(r) = A\,J_n(\lambda r) + B\,Y_n(\lambda r) \tag{28}$$

Here $m^2$ and $\lambda^2$ are called separation constants. Now, isolating the $T$ terms as follows :

$$\frac{1}{-c^2}\frac{1}{T}\frac{d^2T}{dt^2} = k^2 \text{ or}$$

$$\frac{d^2T}{dt^2} + \omega^2 T = 0 \text{ where } \omega^2 = c^2k^2 \tag{29}$$

Its solution is in the form of

$$T(t) = G\,e^{i\omega t} = H\,e^{-i\omega t} \tag{30}$$

The problem is symmetrically loaded so any one term of *cos m*$\theta$ or *sin m*$\theta$ is sufficient to represent the solution of wave equation. So one can safely replace the term $T(t)$ by $e^{i\omega t}$ and $\Theta(\theta)$ by ($\cos m\theta$). So Eq. (21) becomes

$$\phi = Z(z)\,.\,R(r)\,.\ (\cos m\theta)\,e^{i\omega t} \tag{31}$$

where $m$ is the nodal circle and can vary as 0, 1, 2, .... Separation of variables for $z$, using Eq. (23), yields

$$-\frac{Z''}{Z} = k^2 \tag{32}$$

$$\text{where } Z'' = \frac{d^2Z}{dz^2}$$

Eq. (32), yields

$$Z(z) = A\cosh kz + B\sinh kz \tag{33}$$

The physical boundary condition $\frac{\partial\phi}{\partial z}\Big|_{z=0} = 0$ implies that B = 0; therefore

$$Z(z) = A\cosh kz \tag{34}$$

From Eq. (27), one can get

$$R'' + \frac{1}{r} R' + \left[ \frac{\omega^2}{c^2} + k^2 - \frac{m^2}{r^2} \right] R = 0 \qquad (35)$$

Let $\lambda = \left( k^2 + \frac{\omega^2}{c^2} \right)^{1/2}$, then above Eq. reduces to

$$R(r) = C_l J_m(\lambda r) + D_l Y_m(\lambda r) \qquad (36)$$

where $C_1$ and $D_1$ are constants. $D_1 = 0$ since $R$ must be finite when $r \to 0$ (because $Y_m \to \infty$ as $r \to 0$). Thus one can write

$$R = C_l J_m(\beta \bar{r}) \qquad (37)$$

where $\bar{r}$ = r/a and $\beta = a\lambda$ . Since there is no radial component of fluid velocity at $\bar{r} = 1$, for a given value of $m$ there is a set of roots $\beta = \beta_{ms}$(s = 0, 1, 2, 3, ...), where $J'_m(\beta) = 0$. Solving $J'_m(\beta) = 0$, one gets the value of $\beta$. This equation has an infinite number of possible solutions $\beta_{ms}$. To facilitate calculations, let $\pi \beta'_{ms} = \beta_{ms}$. Hence, $\beta_{ms}$ represent the roots of the equation as given by

$$J'_m(\pi \beta'_{ms}) = 0 \qquad (38)$$

Finally, the natural frequency [11] (Hertz) of the fluid can be found as

$$f_{m,s,a} = \frac{c}{2} \left[ \left( \frac{\beta'_{ms}}{a} \right)^2 + \left[ \frac{q}{L} \right]^2 \right]^{1/2} \qquad (39)$$

where $c$ is the acoustic speed in the undisturbed fluid, and the subscripts $m$, $s$ and $q$ are the wave numbers related to the modes of the acoustic oscillations in the closed cylindrical cavity. Here $m$ is the number of nodal diameters, $s$ is the number of nodal circles and $q$ is the number of radial wave in the acoustic media. Table-1 shows the relationship between the Wave numbers and the Acoustical oscillation modes for a closed cylindrical cavity. From Eqs. (31), (34) and (37), one gets

$$\phi = \sum_{m=0}^{+\infty} \sum_{q=1}^{+\infty} \phi_{m,q}(z, \bar{r}, \theta, t) = \sum_{m=0}^{+\infty} \sum_{q=1}^{+\infty} D_{m,q} (\cosh k_{m,q})$$

**Table-1 : Relationship between the wave numbers and the acoustical oscillation modes for a closed cylindrical cavity**

| *Wave number* | | | *Mode of oscillation* |
|---|---|---|---|
| m | s | q | |
| m | 0 | 0 | Tangential |
| 0 | s | 0 | Radial |
| 0 | 0 | q | Longitudinal (axial) |
| m | s | q | Combination |

$$\left[ J_m(\beta_{m,q} r) (\cos m\theta) e^{i\omega t} \right] \qquad (40)$$

where $D_{m,q}$ are unknown constants given by the condition at z = L.

### Coupled Vibration Analysis of Disc Interacting with a Fluid

A circular cylindrical container of radius, $a$, and height, $L$, is filled with a fluid (compressible and non-viscous) of density ρl. The container bottom and its side walls are considered as solid and rigid, while the free fluid surface is covered with an elastic thin circular plate of radius $a$.

It follows from Eq. (40) that for a fixed integer value of $m$ (where $m$ is the number of nodal diameters), the velocity potential of fluid is given [12] as

$$\phi_m(z, \theta, r, t) = \sum_{q=1}^{+\infty} D_{m,q} \cosh(k_{m,q} z) J_m(\beta_{m,q} \frac{r}{a}) \times \cos(m\theta) e^{i\omega t}$$

(41)

and for the forced vibrations of the disc it is possible to write

$$w_m(r, \theta, t) = W_m(r) \cos(m\theta) e^{i\omega t} \qquad (42)$$

where $W_m(r) = \sum_{z=1}^{+\infty} W_{0,m,s} W_{m,s}(r)$ and $W_{0,m,s}$ are unknown constants. Here subscripts $m$ denotes the number of nodal diameter and $s$ denotes the nodal circle. Now velocity in the $z$-direction found from the velocity potential Eq. (41) and by Eq. (42) are equated. This is known as impermeability condition. Here velocities are equated because at the interface where the fluid comes into contact with the circular disc, both the disc and the fluid will have the same velocity. Equating velocities, as obtained by Eqs. (33) and (34), one gets

$$\frac{\partial \phi_m}{\partial z}\bigg|_{z=L} = \frac{\partial w}{\partial t} \tag{43}$$

It gives

$$\sum_{q=1}^{+\infty} D_{m,q}\, k_{m,q} \sinh (k_{m,q}\, l)\, J_m \left(\frac{\beta_{m,q}\, r}{a}\right) = i\,\omega\, W_m (r) \tag{44}$$

Using Orthogonality relationship [10], one gets

$$\int_0^a r\, J_m \left(\beta_{m,q1} \frac{r}{a}\right) J_m \left(\beta_{m,q2} \frac{r}{a}\right) dr = 0 \quad \text{for } q_1 \neq q_2 \tag{45}$$

$$\int_0^a r\, J_m^2 \left(\beta_{m,q} \frac{r}{a}\right) dr$$

$$= \left(\frac{a^2}{2}\right)\left(1 - \frac{m^2}{\beta_{m,q}^{\;2}}\right) J_m^2 (\beta_{m,q}) \quad \text{for } q_1 = q_2 = q \tag{46}$$

Multiplication of Eq. (44) by $\left[ rJ_m\left(\beta_{m,q}\, \frac{r}{a}\right)\right]$ and integrating over the range from 0 to $a$ gives

$$D_{m,q} = \frac{i\omega \int_0^a r\, W_m(r)\, J_m \left(\beta_{m,q} \frac{r}{a}\right) dr}{k_{m,q} \sinh (k_{m,q}\, l)\, J_m^2 \left(\beta_{m,q} \frac{a^2}{2}\right)\left(1 - \frac{m^2}{\beta_{m,q}^2}\right)} \tag{47}$$

On substituting above in Eq. (41) yields

$$\phi_m (z, \theta, r, t)\big|_{z=L} = i\,\omega L \sum_{q=1}^{\infty} \left[ \frac{\coth (k_{m,q} L)}{(k_{m,q} L)} \; \frac{J_m(\beta_{m,q}\, r/a)}{(a^2/2)\,(1 - m^2/\beta_{m,q}^2)\, J_m^2 (\beta_{m,q})} \times \cos(m\,\theta)\, e^{i\omega t} \int_0^a r\, W_m (r)\, J_m (\beta_{m,q} \frac{r}{a})\, dr \right] \tag{48}$$

and pressure of the fluid inside cylindrical container as

$$p_m (r, \theta, z, t)\big|_{z=L} = -\rho_f \frac{\partial \phi_m}{\partial t}\bigg|_{z=L}$$

$$= \rho_f\, L\, \omega^2\, \frac{2}{a^2} \cos (m\,\theta)\, e^{i\omega t} \sum_{q=1}^{+\infty} p_{m,q} (r) \tag{49}$$

where

$$p_{m,q} (r) = \frac{\coth (k_{m,q} L)}{(k_{m,q})} \; \frac{J_m(\beta_{m,q}\, r/a)}{(1 - m^2/\beta_{m,q}^2)\, J_m^2 (\beta_{m,q})}$$

$$\times \int_0^a r\, W_m (r)\, J_m \left(\beta_{m,q} \frac{r}{a}\right) dr \tag{50}$$

Substituting $W_m(r)$ into the integrals of Eq. (50), one gets

$$\int_0^a r\, W_{m,s} (r)\, J_m \left(\beta_{m,q} \frac{r}{a}\right) dr = a^2\, J_m (\beta_{m,q})\, I_m (\beta_{m,s}\, a)\, G_{m,s,q} \tag{51}$$

where

$$G_{m,s,q} = \frac{\beta_{m,s}\, a}{\beta_{m,q}^2 + \beta_{m,s}^2\, a^2} \; \frac{I'_m (\beta_{m,s}\, a)}{I_m (\beta_{m,s}\, a)}$$

$$- \frac{\beta_{m,s}\, a}{\beta_{m,q}^2 - \beta_{m,s}^2\, a^2} \; \frac{J'_m (\beta_{m,s}\, a)}{J_m (\beta_{m,s}\, a)}$$

Therefore the pressure [Eq.(49)] of fluid inside cylindrical container becomes

$$p_m (r, \theta, t)|_{z=L} = 2 (\rho_f L)\, \omega^2 \cos (m\theta)\, e^{i\omega t}$$

$$\times \sum_{q=1}^{+\infty} \frac{\coth (k_{m,q} L)}{(k_{m,q} L)} \times \frac{J_m(\beta_{m,q} r/a)}{(1 - m^2/\beta_{m,q}^2)\, J_m (\beta_{m,q})}$$

$$\times \sum_{s=1}^{+\infty} W_{o,m,s}\, I_m (\beta_{m,s} a)\, G_{m,s,q} \tag{52}$$

where $G_{m,s,q}$ is defined in Eq. (51) above. The pressure value obtained by Eq. (52) has to be supplied as boundary conditions for disc vibrating at the surface of the fluid. For the disc in vacuo, the equation of motion of the disc is given by

$$\nabla^4 W_{m,s}(r) = \frac{\rho_d}{D_0} \omega_{m,s}^2 W_{m,s}(r) \tag{53}$$

and the equation of motion, describing the vibrating disc interacting with the fluid, is given [13] as

$$\nabla^4 w = -\frac{\rho_d}{D_0} \frac{\partial^2 w}{\partial t^2} + \frac{p}{D}\Big|_{z=L}$$

Substitutions of disc deflection i.e., $w_m (r, \theta, t) = \cos (m\theta)\, e^{i\omega t} \sum_{s=1}^{+\infty} W_{0,m,s}\, W_{m,s}(r)$ and $p_m$ at z = L in the above equation yields

$$\frac{\rho_d}{D_0} \sum_{s=1}^{+\infty} W_{0,m,s} (\omega_{m,s}^2 - \omega^2)\, W_{m,s}(r) = \frac{p_m}{D}\Big|_{z=L} \tag{54}$$

After substitution of $W_{m,s}$ from Eq. (17) and $p_m$ from Eq. (52) in Eq. (54), one gets

$$\sum_{s=1}^{+\infty} \left(\frac{\rho_d}{D_0}\right) (\omega_{m,s}^2 - \omega^2)\; W_{0,m,s}$$

$$\times \left[ -\frac{I_m (\xi_{m,s})}{J_m (\xi_{m,s})} J_m \left(\xi_{m,s} \frac{r}{a}\right) + I_m \left(\xi_{m,s} \frac{r}{a}\right) \right]$$

$$= -2 \left(\frac{\rho_f\, l}{D_0\, h}\right) \omega^2 \sum_{s=1}^{+\infty} W_{0,m,s}\, I_m (\xi_{m,s}) \sum_{q=1}^{+\infty} C_{m,q}(\omega)\, G_{m,s,q}$$

$$\times \frac{J_m (\beta_{m,q} r/a)}{(1 - m^2/\beta_{m,q}^2)\, J_m (\beta_{m,q})} \tag{55}$$

where, $C_{m,q}(\omega) = -\dfrac{\coth (k_{m,q}\, l)}{(k_{m,q}\, l)}$. Now multiplying Eq. (55) with $\left[ r\, J_m (\beta_{m,q} r/a) \right]$ and then integrating over the interval (0, a), one gets

$$\sum_{s=1}^{+\infty} (\omega_{m,s}^2 - \omega^2)\; W_{o,m,s} \int_0^a r\, J_m (\beta_{m,q} r/a)$$

$$\times \left[ -\frac{I_m (\xi_{m,s})}{J_m (\xi_{m,s})} J_m \left(\xi_{m,s} \frac{r}{a}\right) + I_m \left(\xi_{m,s} \frac{r}{a}\right) \right] dr$$

$$= -2\, \rho\, \omega^2 \sum_{s=1}^{+\infty} W_{0,m,s}\, I_m (\xi_{m,s}) \sum_{q=1}^{+\infty} C_{m,q}(\omega)\, G_{m,s,q}$$

$$\times \frac{1}{(1 - m^2/\beta_{m,q}^2)\, J_m (\beta_{m,q})}$$

$$\times \int_0^a r\, J_m (\beta_{m,q} r/a)\, J_m (\beta_{m,q} r/a)\, dr \tag{56}$$

where $\rho = \rho_f L/\rho_d h$. Finally performing the integration of above equation, one gets,

$$\sum_{s=1}^{+\infty} (\omega_{m,s}^2 - \omega^2)\, W_{o,m,s}\, I_m (\beta_{m,s} a)\, G_{m,s,q}\, J_m (\beta_{m,q})$$

$$= -\rho\, \omega^2 \sum_{s=1}^{+\infty} W_{o,m,s}\, I_m (\beta_{m,s} a)\, C_{m,q}(\omega)\, G_{m,s,q}\, J_m (\beta_{m,q})$$

which after rearrangement gives

$$\sum_{s=1}^{+\infty} W_{o,m,s}\, I_m (\beta_{m,s} a) \left[ (\omega_{m,s}^2 - \omega^2)\, G_{m,s,q}\, J_m (\beta_{m,q}) \right.$$

$$+ \rho \omega^2 C_{m,q} (\omega) G_{m,s,q} J_m (\beta_{m,q}) \Big] = 0 \qquad (57)$$

Expressing Eq. (57), in matrix form for fixed *m* and *s* (as 1, 2, ... n), gives us the values of different frequencies by equating its determinant to zero. When the matrix, thus, formed is of the order of 3 or more the solution of mathematical formulation becomes very complex so ANSYS have been used to calculate the coupled frequencies.

To get the solution of mathematical formulations presented above, radius of the circular disc is taken as 0.038m, thickness of disc as 0.00038m, length of cylinder as 0.081m, density of fluid inside the cylinder as 1.2 Kg/$m^3$, density of circular disc as 7800 kg/$m^3$ and sonic velocity as 343 m/s.

### ANSYS Modeling and Results

The finite element modeling of a thin circular disc clamped at its periphery is made on ANSYS-10[14]. For this purpose, a solid-shell 3-D finite strain 190 element is used for the finite element modeling. The frequencies of the disc as obtained by ANSYS-10 are shown in Table-2. Theoretical frequencies of the disc have also been obtained from frequency equation i.e. Eq. (14) for few nodal diameter (m) and nodal circle (n). For these nodal diameter and nodal circle, quantitative comparison of the frequencies as obtained by the above two manner is shown in Table-3. The Table-3 shows that the frequencies are in close agreement to each other.

**Table-2 : Frequencies of circular disc clamped at its periphery by ANSYS-10**

| *Set No.* | *Frequency (Hz)* |
|---|---|
| 1 | 671.88 |
| 2 | 1398.5 |
| 3 | 1398.6 |
| 4 | 2294.6 |
| 5 | 2294.7 |
| 6 | 2616.7 |
| 7 | 3358.0 |
| 8 | 3358.1 |
| 9 | 4003.9 |
| 10 | 4004.3 |
| 11 | 4585.3 |
| 12 | 4585.6 |
| 13 | 5569.4 |
| 14 | 5570.4 |
| 15 | 5866.6 |

The finite element modelling for the fluid alone is then carried out on ANSYS-10. For this, FLUID 30 element is chosen for the fluid and SOLID 95 element for the cylinder walls containing the fluid. The frequencies of fluid inside a closed cylindrical container are obtained by ANSYS and are shown in Table-4. Theoretical frequencies of the fluid alone have also been obtained from frequency equation i.e. Eq. (39) for few nodal diameter (m), nodal circle (n) and number of radial wave in the acoustic media (q). For these nodal solution, quantitative comparison of the frequencies as obtained by the above two methods is shown in Table-5. It is clear from Table-5 that results of ANSYS modelling are in close agreement with the theoretical one.

Finally, the finite element modeling of complete problem of disc interacting with the fluid contained inside the cylindrical container is analyzed by ANSYS -10. For this purpose, fluid is assumed to be compressible and non-viscous fluid. SHELL63 element is selected for disc and thickness of the disc is taken as 0.00038m. FLUID 30 element is selected for fluid and SOLID 95 element for container walls from the ANSYS element library. The frequencies of the circular disc interacting with a fluid contained inside the cylindrical container as obtain by ANSYS-10 is shown in Table-6. Quantitative comparison of the ANSYS results with the theoretical values, as obtained by Eq. (57), is made in Table-7 for few nodal solutions. The Table-7 shows that the percentage error in

**Table-3 : Quantative comparison of frequencies of theoretical and ANSYS methods for circular disc alone for few nodal solutions**

| *S. No.* | *m, n* | *Theoretical Frequency (Hz)* | *ANSYS-10 Frequency (Hz)* |
|---|---|---|---|
| 1 | 0, 0 | 671.50 | 671.88 |
| 2 | 1, 0 | 1398.00 | 1398.5 |
| 3 | 2, 0 | 2292.50 | 2294.6 |
| 4 | 0, 1 | 2615.71 | 2616.7 |
| 5 | 3, 0 | 3358.30 | 3358.0 |
| 6 | 1, 1 | 4001.00 | 4003.9 |
| 7 | 0, 2 | 5859.83 | 5866.6 |

**Table-4 : Frequencies of fluid alone inside a closed cylindrical container by ANSYS-10 assuming container has rigid walls**

| Set. No. | Frequency (Hz) |
|---|---|
| 1 | 2120.8 |
| 2 | 3397.7 |
| 3 | 3401.5 |
| 4 | 4262.3 |
| 5 | 4902.3 |
| 6 | 4909.8 |
| 7 | 5021.4 |
| 8 | 5024.0 |
| 9 | 5979.6 |
| 10 | 6146.3 |
| 11 | 6280.3 |
| 12 | 6445.2 |
| 13 | 6483.6 |
| 14 | 7027.2 |
| 15 | 7370.6 |

**Table-5 : Quantative comparison of frequencies as obtained by Theoretical and ANSYS methods for fluid alone for few nodal solutions**

| *S. No* | *m, n, q* | *Theoretical Frequency (Hz)* | *ANSYS Frequency (Hz)* | *% Error* |
|---|---|---|---|---|
| 1 | 0, 0, 1 | 2116.50 | 2120.8 | 0.2 |
| 2 | 0, 0, 3 | 3386.54 | 3397.7 | 0.3 |
| 3 | 1, 0, 0 | 4236.60 | 4262.3 | 0.6 |
| 4 | 2, 0, 0 | 4874.30 | 4902.3 | 0.6 |
| 5 | 3, 0, 0 | 6034.09 | 6146.3 | 1.8 |

these two results is within 2.85 % and are in close agreements.

A qualitative nodal solution, for circular disc alone which is clamped at its outer periphery, obtained by ANSYS-10, is then shown in Fig.3 for various frequencies. The Fig.4 shows the qualitative nodal solutions of a circular disc interacting with a fluid contained inside the cylindrical vessel, as obtained by ANSYS-10 for various frequencies.

**Table-6 : Frequencies of vibration of a circular disc interacting with a fluid contained inside the cylindrical container by ANSYS-10**

| *Set No.* | *Frequency (Hz)* |
|---|---|
| 1 | 671.39 |
| 2 | 1393.0 |
| 3 | 1395.2 |
| 4 | 2120.6 |
| 5 | 2281.1 |
| 6 | 2286.1 |
| 7 | 2604.7 |
| 8 | 2654.4 |
| 9 | 2654.4 |
| 10 | 3333.5 |
| 11 | 3341.2 |
| 12 | 3397.6 |
| 13 | 3599.2 |
| 14 | 3964.8 |
| 15 | 3980.4 |
| 16 | 4268.0 |
| 17 | 4545.0 |
| 18 | 4550.0 |
| 19 | 5036.8 |
| 20 | 5518.5 |

**Table-7 : Quantative comparison of frequencies obtained by theoretical and ANSYS methods for circular disc interacting with fluid inside cylindrical container for few nodal solutions. Here m is the number of nodal diameters, s is the number of nodal circles and q is the number of radial wave in the acoustic media**

| *S. No.* | *m, n, q* | *Theoretical Frequency (Hz)* | *ANSYS Frequency (Hz)* | *% Error* |
|---|---|---|---|---|
| 1 | 0, 0, 0 | 675.1 | 671.39 | 0.5 |
| 2 | 1, 0, 0 | 1393.4 | 1393.0 | 0.02 |
| 3 | 2, 0, 0 | 2294 | 2286.1 | 0.34 |
| 4 | 0, 0, 1 | 2131.1 | 2120.6 | 0.49 |
| 5 | 0, 1, 0 | 2615 | 2604.7 | 0.39 |
| 6 | 1, 0, 0 | 2644.2 | 2654.4 | 0.38 |
| 7 | 1, 0, 1 | 3387 | 3397.6 | 0.31 |
| 8 | 2, 0, 0 | 4390 | 4268.0 | 2.85 |

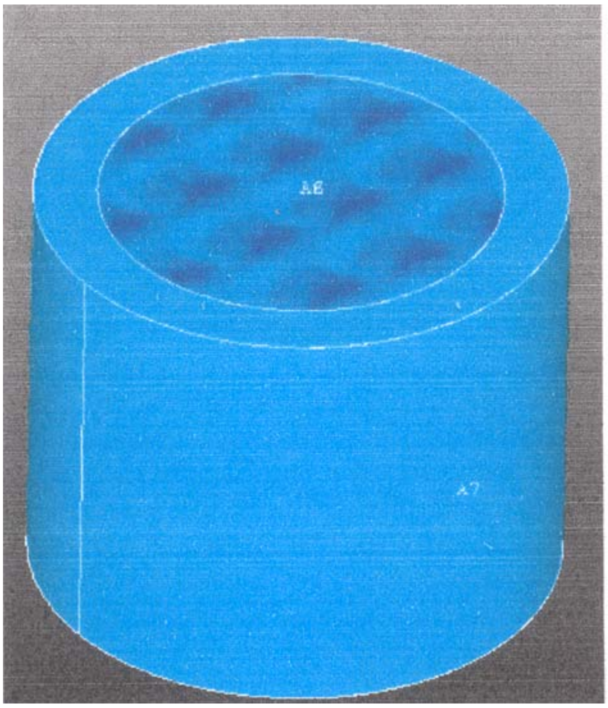


*Fig.2 Cylindrical acoustic cavity*

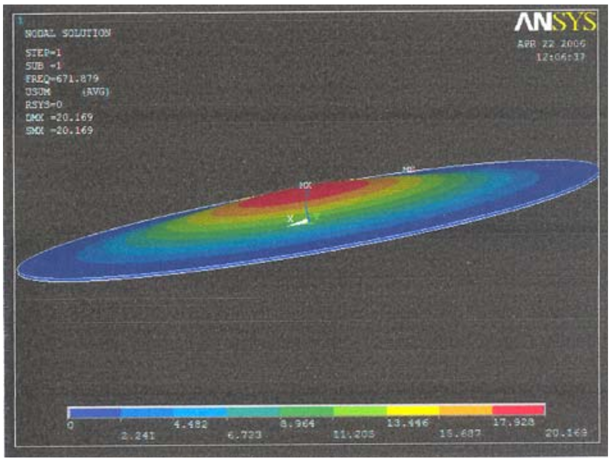


*Fig.3a Plots for the deformed shapes (Nodal solutions) at frequency = 671.879 Hz*

**Discussions and Conclusions**

Vibration analysis of a thin circular disc interacting with a fluid in cylindrical container is carried out in this paper. For this, theoretical modeling of a circular disc clamped at its periphery, fluid contained in a cylindrical vessel and the circular disc interacting with the fluid contained by a cylindrical vessel is presented. The above

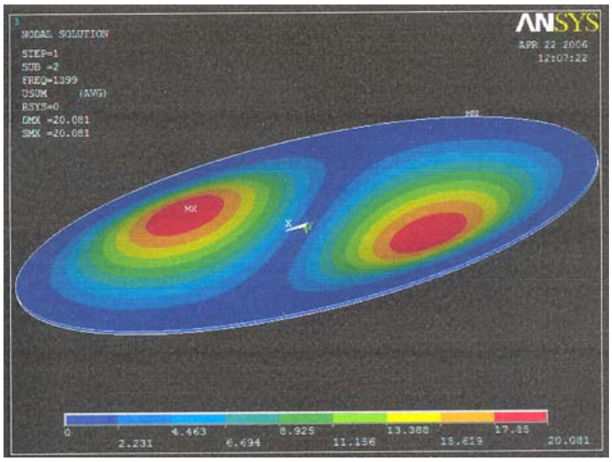


*Fig.3b Plots for the deformed shapes (Nodal solutions) at frequency = 1399 Hz*

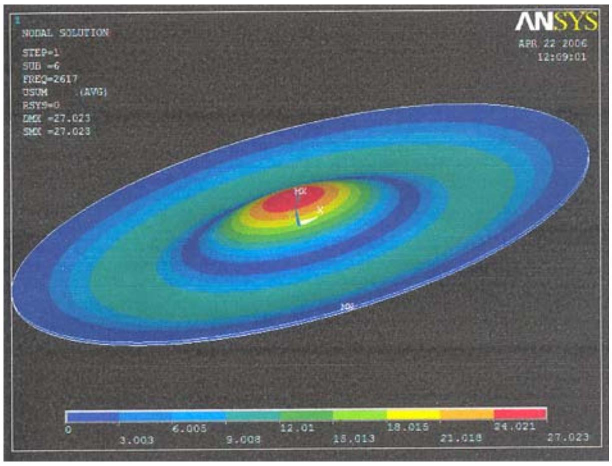


*Fig.3c Plots for the deformed shapes (Nodal solutions) at frequency = 2617 Hz*

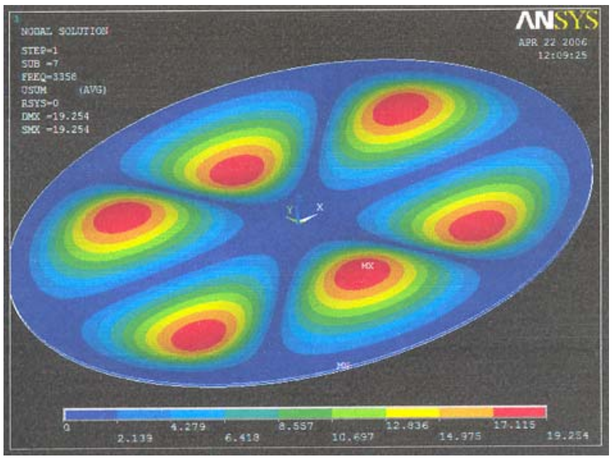


*Fig.3d Plots for the deformed shapes (Nodal solutions) at frequency = 3358 Hz*

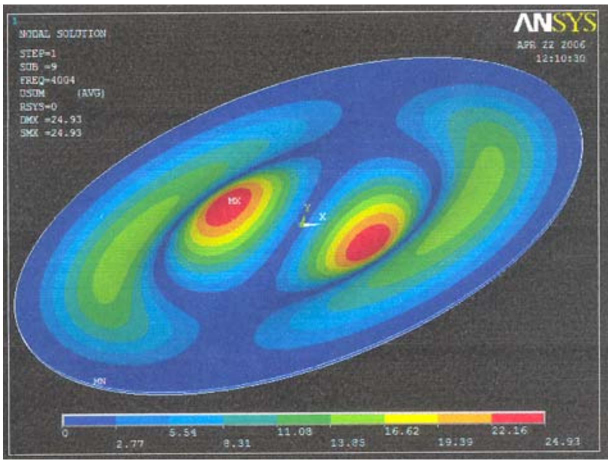


*Fig.3e Plots for the deformed shapes (Nodal solutions) at frequency = 4004 Hz*

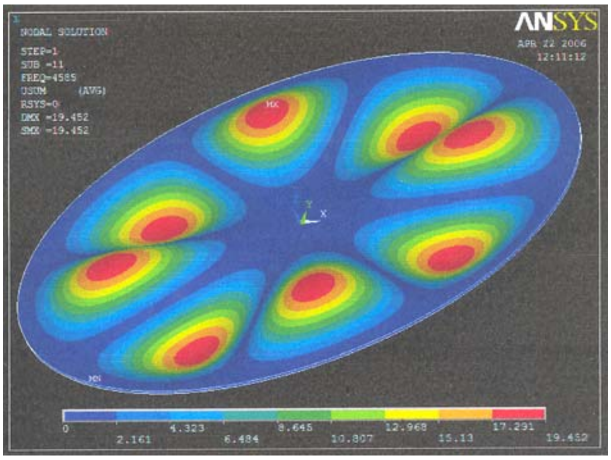


*Fig.3f Plots for the deformed shapes (Nodal solutions) at frequency = 4585 Hz*

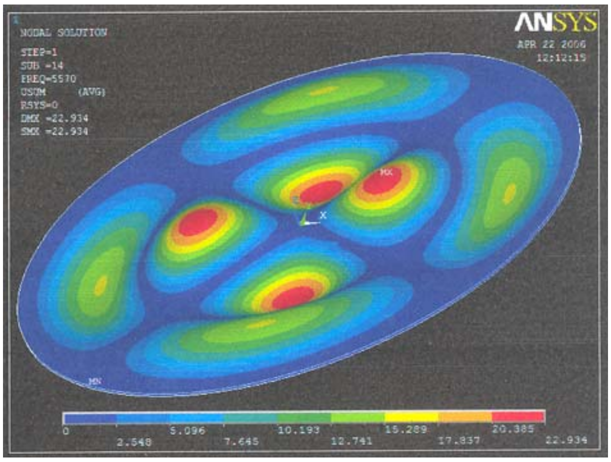


*Fig.3g Plots for the deformed shapes (Nodal solutions) at frequency = 5570 Hz*

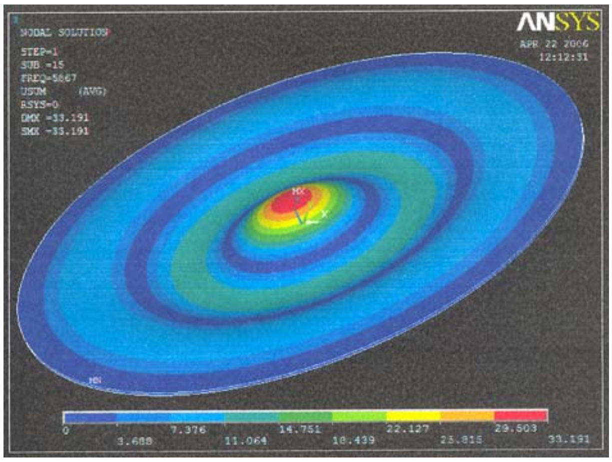


*Fig.3h Plots for the deformed shapes (Nodal solutions) at frequency = 5870 Hz*

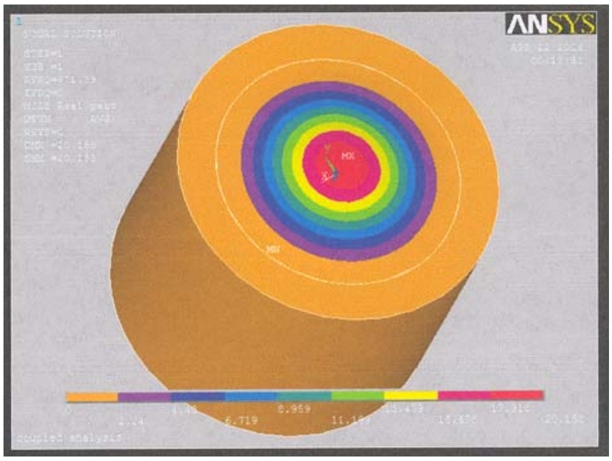


*Fig.4a Nodal solutions for the coupled case at 671.39 Hz*

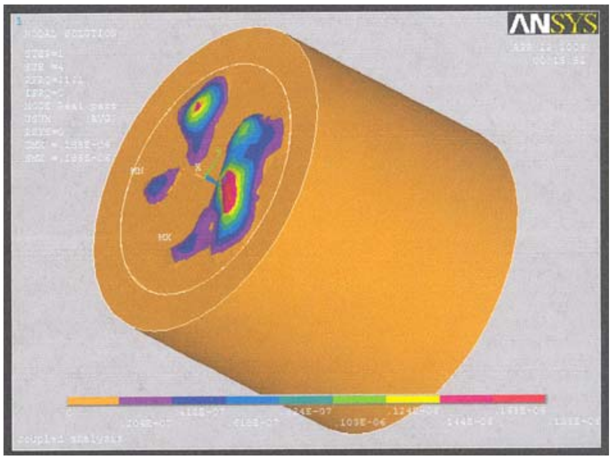


*Fig.4b Nodal solutions for the coupled case at 1393 Hz*

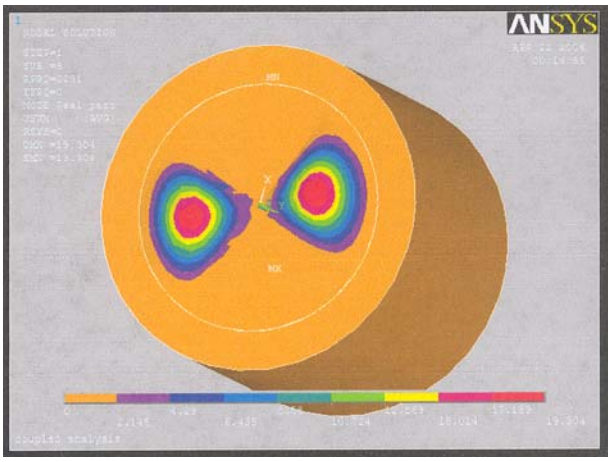

*Fig.4c Nodal solutions for the coupled case at 2121 Hz*

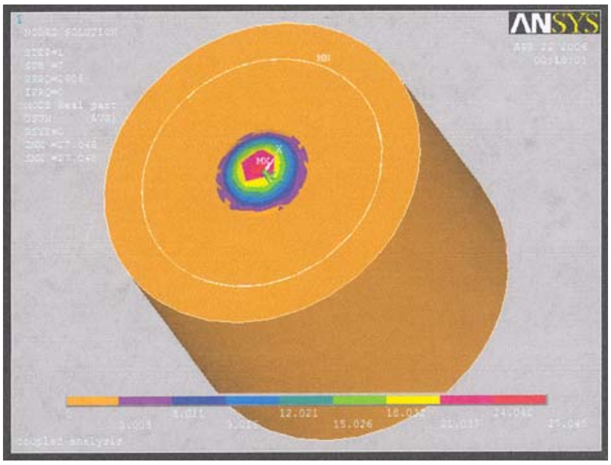

*Fig.4d Nodal solutions for the coupled case at 2281 Hz*

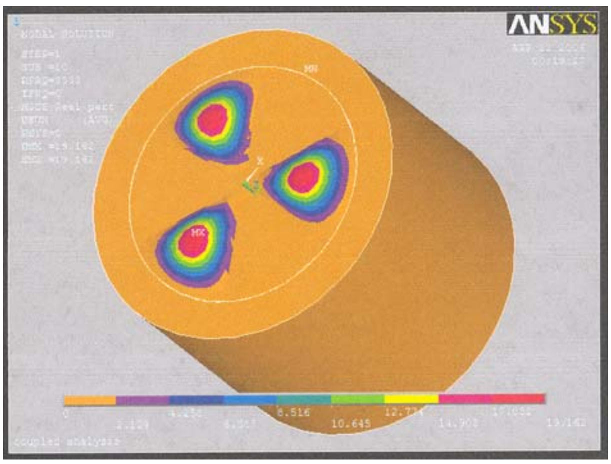

*Fig.4e Nodal solutions for the coupled case at 2605 Hz*

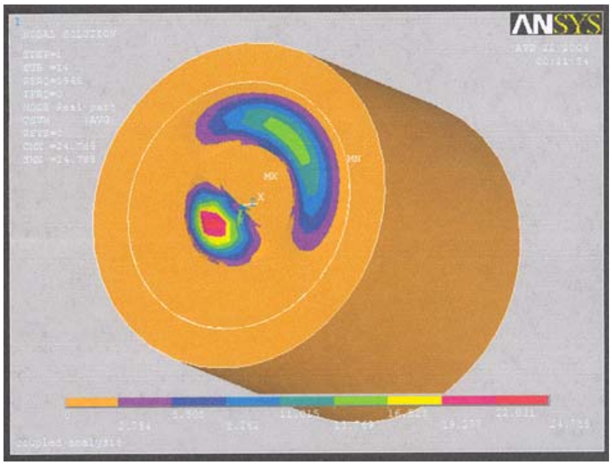

*Fig.4f Nodal solutions for the coupled case at 3333 Hz*

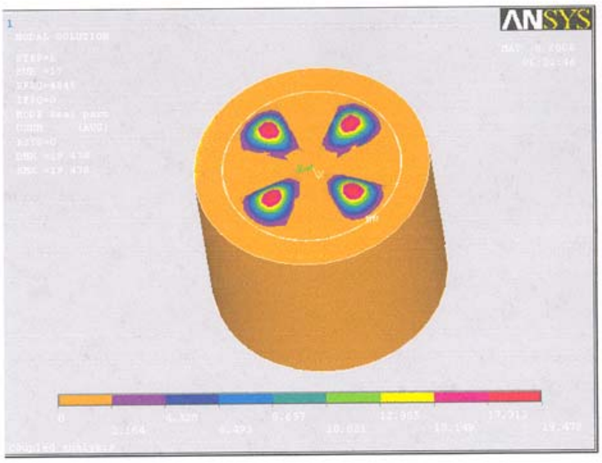

*Fig.4g Nodal solutions for the coupled case at 3965 Hz*

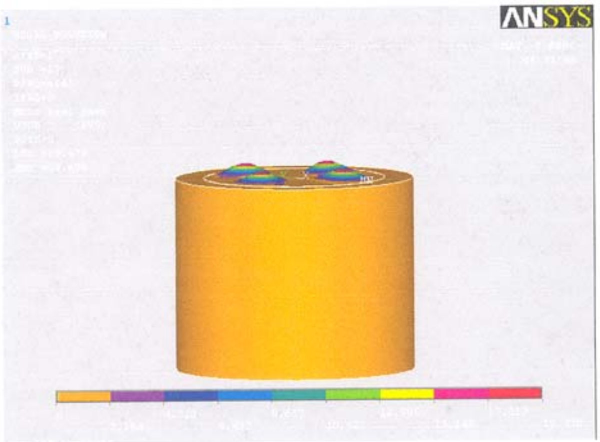

*Fig.4g Nodal solutions for the coupled case at 4545 Hz (two different views*

three theoretical modeling is compared quantatively for few nodal solutions by ANSYS-10 modeling. The results are found to be in close agreement for the two methods. Qualitative nodal solutions are, then, presented for the circular disc alone and the circular disc interacting with the fluid in side a cylindrical vessel as obtained by ANSYS-10.

For a circular disc interacting with fluid inside cylindrical container, It has been observed that for length of 0.081m, only few values of coupled analysis show strong fluid-structure results, while the other frequencies obtained are either very near to the disc frequencies or fluid frequencies alone. But if one increase the length of the cylinder, it is possible to show that more and more frequencies have strong fluid-structure interaction effects. So by studying the coupled vibration analysis, one can predict the natural frequencies within which the coupled system (consisting of the disc and the fluid) will vibrate safely and so it can help designer in knowing the frequencies at which this system might come in resonance with some other machinery with which it might be attached or coming in contact.

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