FULL LENGTH PAPER

A SIMPLE, CORRECT PEDAGOGICAL PRESENTATION OF AIRPLANE LONGITUDINAL DYNAMICS

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Abstract

Various shortcomings and inaccuracies in the existing approach and presentation of airplane flight dynamics in pedagogy have been identified. An improved pedagogical presentation is offered which is clean - uses the two timescales in longitudinal dynamics to derive short period and phugoid mode parameters, simple - starts with the longitudinal dynamics equations rather than the complete six degree of freedom formulation, and correct - uses a physically correct model of the dynamic (rate) derivatives and employs the concept of a static residual in deriving the slower (phugoid) mode approximations. The redundant and confusing concepts of dimensional derivatives and static stability are junked. A companion paper presents a similar approach to the lateral-directional dynamics.

Introduction

Beginning with the early work by Bryan [1], the equations of rigid airplane flight dynamics, the analysis of airplane small perturbations about a trim state, and the modeling of aerodynamic forces under small perturbations have been passed down and followed virtually unchanged for nearly a century. The approach, analysis procedure and the results therefrom have been enshrined in classic flight dynamics textbooks such as those by Perkins and Hage [2], Etkin [3], Seckel [4] and Roskam [5], which have been used to teach several generations of students in the classroom. The same material is found in texts with an emphasis on flight control and simulation, for example, Blakelock [6], McRuer, Ashkenas and Graham [7], McLean [8], Stevens and Lewis [9] and Nelson [10]. A number of other textbooks have since appeared on the subject, such as Pamadi [11], Phillips [12] and Stengel [13]. Without exception, all of them contain the same material in roughly the same sequence - definition of various axes and transformations between them, derivation of the 6-degree of freedom equations of motion for a rigid airplane, selection of a trim state, usually straight and level flight, small perturbations and the derivation of linearized equations about the chosen trim state, decoupling of the linearized dynamics into lateral and longitudinal sets, reduction of the decoupled dynamics into the airplane modes and a discussion of stability (termed *dynamic stability*) based on the eigenvalues (equivalently, the frequency and damping, or the time-to-half/time-to-double amplitude) of the various modes (short period, phugoid, roll, dutch roll, and spiral). Separately, a notion of *static stability* is introduced which leads to a discussion of the three aerodynamic derivatives - $C_{m\alpha}$ (longitudinal static stability), $C_{n\beta}$ (directional static stability), and $C_{l\beta}$ (lateral static stability), and subsequently to elevator, rudder and aileron control.

Over the years, it has been widely recognized that this traditional approach and presentation has some technical as well as pedagogical shortcomings.

- Students usually find the axes transformations and the derivation of the 6-degree of freedom equations of motion dreary and complex. This causes some of them to turn off the subject early in the course. Oddly enough, the 6-degree of freedom equations are then mostly used only to study longitudinal and lateral small-perturbation dynamics about a straight and level flight trim - something that could have been done without all the labor and time invested in and the dislike

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engendered by the derivation of the complete equations of motion.

- The small-perturbation aerodynamic model is first written in terms of the *dimensional aerodynamic derivatives*. Later, the dimensional derivatives are expressed in terms of *non-dimensional aerodynamic derivatives*, which are more natural aerodynamic quantities. For instance, the non-dimensional $C_{L\alpha}$ is more natural and understandable than the dimensional Z_w. It makes sense to discard the dimensional derivatives and directly write the aerodynamic model in terms of the non-dimensional derivatives. There is also a more serious error that has crept in due to the use of the dimensional derivatives, as we see next.
- The dimensional derivatives were introduced by Bryan [1] in 1911 on purely mathematical grounds, without regard to the aerodynamics. Later the focus was on representing these dimensional derivatives correctly in terms of the non-dimensional aerodynamic derivatives. The definition of the dimensional derivatives itself became a *fait accompli* and was never questioned until recently (Ananthkrishnan [14], and Raghavan and Ananthkrishnan [15]). It turns out the rate derivative (also called dynamic derivative) terms as they are traditionally represented in the aerodynamic model do not agree with standard aerodynamic theory. There ought to be two terms - one depending on the relative angular velocity of the airplane with respect to the wind, and the second a function of the wind angular velocity (also called *flow curvature effect* [21]). Instead, the traditional aerodynamic model only provides for a single dimensional derivative (along each axis). Even when the dimensional derivative is transcribed correctly in terms of the non-dimensional aerodynamic derivative, it is multiplied in the model by the wrong variable - airplane angular velocity instead of the difference between the airplane and wind angular velocities. This leads to a grotesque situation as for example below. Consider the derivative C_{mq}, also called the pitch damping derivative. It is traditionally modeled as $C_{mq}\Delta q_b$, where Δq_b is the perturbed airplane pitch rate. One may equivalently write this term as follows, where Δq_w is the velocity vector (wind axis) angular rate in pitch: $C_{mq}\,\Delta q_b = C_{mq}\,(\Delta q_b - \Delta q_w) + C_{mq}\,\Delta q_w$. The first term on the right-hand side is the correct relative angular velocity effect and the second one is an unintentional and incorrect flow curvature effect that appears irrespective of the airplane configuration. In fact, the flow curvature effect in pitching motion is usually negligible except in special cases such as airplanes with T-tails. The incorrect second term needlessly couples the relative-angular-velocity-dominant mode (short period) with the flow-curvature-dominant mode (phugoid) and introduces errors in both their models. Likewise for the relative-angular-velocity-dominant dutch roll mode and its counterpart flow-curvature-dominant spiral mode in the lateral-directional dynamics. Yet this century-old error continues to permeate every textbook in the market today.
- The notion of static stability, perhaps unique to the field of flight dynamics, is the cause of much confusion. Possibly in an age when the theory of stability of dynamic systems was not widespread among engineers and analytical methods were the only recourse, one could have justified the use of an approximation such as static stability. To carry it forward to the 21st century is a bit ludicrous. No textbook connects up the concepts of static and dynamic stability, possibly because they are not always compatible. For example, under the traditional model, it is possible for a mode to be statically unstable and yet dynamically stable - a feat impossible in the annals of mathematics, yet one that generations of flight dynamicists have been handwaving away. The discrepancy can be resolved to some extent by fixing the error in the aerodynamic model discussed immediately above. However, the saner option is to drop the notion of static stability entirely and stick to a single concept of (dynamic) stability, common with other scientific disciplines.
- The process of obtaining literal approximations to the various airplane dynamic modes, and thence their frequency and damping, or time-to-half/time-to-double, causes some disquiet. The derivation is frankly *ad hoc* and further most authors admit that the approximations to the phugoid, dutch roll and spiral modes are poor and not to be relied upon. This could partly be due to the error in the aerodynamic model as discussed above. However, it was shown by Ananthkrishnan and Unnikrishnan [16], and Ananthkrishnan and Ramadevi [17] that, subject to the aerodynamic model being correct, accurate literal approximations to the flight dynamic modes could be obtained by following a proper multiple timescale approach. The key is to recognize that a static residual of the faster mode participates in the dynamics at the slower timescale. While the existence of disparate timescales between the modes has been generally acknowledged, no textbook explicitly writes the timescales down nor uses them to separate the different modes. As for the corrected literal approxima-

tions, no textbook bar one (Stevens and Lewis [9]) has bothered to incorporate them over the past decade.

- Lastly, in this age of computation, students must be introduced to the computational tools available and used in industry. The standard tool for flight dynamic analysis today is the bifurcation and continuation method. While bifurcation methods originally entered the picture as a specialized nonlinear, especially high angle of attack, analysis tool, today they may be used for flight performance and stability analysis across all segments of the flight envelop (Ananthkrishnan and Sinha [18], Paranjape, Ananthkrishnan and Sinha [19]). In fact, flight dynamics has been an early and notably successful application area of bifurcation theory. The introduction of bifurcation methods with real-life airplane data will provide students with a flavor of industrial practice as well as expose them to life beyond the sanitized, linear domain that textbooks must perforce focus on. For instance, many modern military airplanes stall around 30-40 deg angle of attack, not at the 15-18 deg range that is touted by textbooks.

The present paper is the outcome of the authors' experience in teaching flight dynamics for over two decades. It is our belief that the traditional presentation and contents of a course in flight dynamics must be overhauled and improved in keeping with recent revelations and the requirement to tie up the theory with real-life examples and industrial practice. While this paper focuses on the longitudinal dynamics, a companion paper presents our approach to the lateral-directional dynamics. The papers form the basis for a new textbook on flight dynamics that is in press (Sinha and Ananthkrishnan [20]).

Equations of Motion in the Longitudinal Plane

The motion of an airplane in flight is described by two vectors as shown in Fig.1 - the velocity **V** of its center of gravity (cg) and its angular velocity vector ω. By placing a set of axes $\mathbf{X^B Y^B Z^B}$ fixed to the airplane body at its cg, **V** is the velocity of the origin of this body-fixed axis system with respect to the Earth-fixed inertial axes $\mathbf{X^E Y^E Z^E}$, and ω is the angular velocity of the body-fixed axes with respect to the Earth-fixed axes. The airplane in flight has six degrees of freedom, corresponding to the three components of the velocity **V** and the three components of the angular velocity ω.

The longitudinal plane defined by the axes $\mathbf{X^B}$-$\mathbf{Z^B}$ is usually an axis of symmetry for most airplanes. Additionally, a significant part of airplane flight happens with the velocity vector **V** lying in the longitudinal plane. The dominant aerodynamic force, the lift, lies in the longitudinal plane and most maneuvers arise by manipulating the lift vector. Naturally, flight in the longitudinal plane gets much attention. Pedagogically, it is easier to introduce various concepts related to airplane dynamics and aerodynamic modeling by first considering the case of longitudinal flight.

Consider an airplane flying an arbitrary curved trajectory in the longitudinal plane as shown in Fig.2. The various angles and forces acting on the airplane are also marked therein. The equations of motion in the longitudinal plane can be written without further ado by straightforward appeal to the laws of physics.

$$m \frac{dV}{dt} = T\cos(\theta - \gamma) - D - W\sin\gamma \tag{1a}$$

$$mV\frac{d\gamma}{dt} = T\sin(\theta - \gamma) + L - W\cos\gamma \tag{1b}$$

$$I_{yy}\frac{dq}{dt} = M \tag{1c}$$

where V is the scalar total velocity, γ is the flight path angle, q is the body-axis pitch rate, and θ is the body-axis pitch Euler angle. L, D, M, W, T are respectively the lift, drag, pitching moment, weight and thrust. m is the airplane mass and I_{yy} its moment of inertia about the $\mathbf{Y^B}$ axis. Equation (1) is supplemented by the kinematic relations:

$$\dot{x} = V\cos\gamma \tag{2a}$$

$$\dot{z} = -V\sin\gamma \tag{2b}$$

$$\dot{\theta} = q \tag{2c}$$

where x and z are the distance measured along the Earth-fixed axes $\mathbf{X^E}$ and $\mathbf{Z^E}$ respectively.

The obvious next step is to write the aerodynamic forces and moment in Eq.(1) from aerodynamic theory. The usual expressions for the drag, lift and pitching moment in terms of their respective coefficients C_D, C_L, C_m are : $D = \bar{q}SC_D$, $L = \bar{q}SC_L$, and $M = \bar{q}Sc\,C_m$, where $\bar{q}$ is the dynamic pressure (= 1/2 ρ $\mathbf{V}^2$), S is a reference area, usually the airplane wing planform area, and c is the mean aerodynamic chord. Then Eq.(1) appears as:

$$m\frac{dV}{dt} = T\cos(\theta-\gamma) - \bar{q}SC_D - W\sin\gamma \quad (3a)$$

$$mV\frac{d\gamma}{dt} = T\sin(\theta-\gamma) + \bar{q}SC_L - W\cos\gamma \quad (3b)$$

$$I_{yy}\frac{dq}{dt} = \bar{q}ScC_m \quad (3c)$$

Timescales

Equation (3) with Eq.(3c) combined with Eq.(2c) can be written as follows:

$$\frac{\dot{V}}{V} = \left(\frac{g}{V}\right)\left[\left(\frac{T}{W}\right)\cos(\theta-\gamma) - (\bar{q}\,S/W)\,C_D - \sin\gamma\right] \quad (4a)$$

$$\dot{\gamma} = \left(\frac{g}{V}\right)\left[\left(\frac{T}{W}\right)\sin(\theta-\gamma) + (\bar{q}\,S/W)\,C_L - \cos\gamma\right] \quad (4b)$$

$$\ddot{\theta} = (\bar{q}\,Sc/I_{yy})\,C_m \quad (4c)$$

where two timescales emerge naturally. The faster of these is:

$$T_1 = \sqrt{\frac{I_{yy}}{\bar{q}Sc}} \quad (5a)$$

called the pitch timescale, which for most conventional airplanes is of the order of 1 second. This corresponds to the pitching motion, i.e., nose bobbing up and down, and represents the rate of change of θ. The slower timescale is:

$$T_2 = \left(\frac{V}{g}\right) \quad (5b)$$

For most airplanes, this is of the order of 10 seconds and corresponds to the heaving motion, i.e., the airplane alternately gains and loses altitude (or, climbs and descends). The variables V and γ naturally change at this rate. It can be seen that T_2 is one order (that is, around 10 times) slower than the faster timescale T_1. A key physical rule is that phenomena that occur at clearly distinct timescales can be studied separately. It is this very rule that allows us to investigate the pitch dynamics (T_1 timescale) independently of the heave dynamics (T_2 timescale) because they are so well separated. For illustration, Fig.3 shows a heave variable changing over a time period of 10 sec; superimposed on it is the oscillation of the pitch variable over a time period of 1 sec. During the time the pitch variable is active, the heave variable changes little. Hence, one is justified in assuming that the heave variables maintain constant values when the pitch dynamics is ongoing. Contrariwise, when studying several cycles of the slower heave dynamics, any pitch activity appears as a brief blip that quickly damps out and may hence be ignored. This last statement needs to be slightly qualified, as we point out later in this paper.

Trim States

Trim states in longitudinal flight are obtained by setting the left-hand side of Eq.(4) to zero. Thus, solving

$$\left[\left(\frac{T}{W}\right)\cos(\theta^* - \gamma^*) - (\bar{q}^*\,S/W)\,C_D^* - \sin\gamma^*\right] = 0 \quad (6a)$$

$$\left[\left(\frac{T}{W}\right)\sin(\theta^* - \gamma^*) + (\bar{q}^*\,S/W)\,C_L^* - \cos\gamma^*\right] = 0 \quad (6b)$$

$$C_m^* = 0 \quad (6c)$$

yields the trim values of the variables - V^*, γ^*, θ^*, where the superscript '*' signifies a trim value. Usual trim states are straight and level/ascending/descending flight. Since airplane dynamic behavior in shallow ascents/descents is not very different from that in level flight, it is normal to concentrate only on straight and level flight trims.

Pitch/ Short Period Dynamics

Following the timescale separation argument above, we are justified in considering Eq.(4c) alone for the pitch dynamics while V, γ are held fixed at their trim values - V^*, γ^*. Writing small perturbations about a trim state, we have

$$\ddot{\theta} + \Delta\ddot{\theta} = \left(\frac{\bar{q}\,Sc}{I_{yy}}\right)(C_m^* + \Delta C_m) \quad (7a)$$

And since $\ddot{\theta} = 0$ at trim and $C_m^* = 0$ as well from Eq. (6c), we obtain from Eq. (7a),

$$\Delta\ddot{\theta} = \left(\frac{\bar{q}\,Sc}{I_{yy}}\right)\Delta C_m \quad (7b)$$

as the small-perturbation equation in pitch. Referring to Fig.2, we have the following relation between the angles: $\theta = \alpha + \gamma$. Hence at trim, $\theta^* = \alpha^* + \gamma^*$. Under small perturbation, since γ remains undisturbed at γ^*, we have $\Delta\theta = \Delta\alpha$. Thus, one may write the pitch perturbation dynamics in Eq.(7b) as,

$$\Delta\ddot{\alpha} = \left(\frac{\bar{q}\,Sc}{I_{yy}}\right)\Delta C_m \tag{7c}$$

which brings us to the crucial question of modeling the small-perturbation aerodynamics for ΔC_m.

Small-perturbation Aerodynamic Modeling (Pitch Motion)

The aerodynamic model for the perturbed coefficients such as ΔC_m must be based on known facts from aerodynamic theory and not be railroaded into a functional form depending on the choice of variables used to represent the flight dynamic states.

The aerodynamic forces acting on an airplane arise due to its interaction with the relative wind. Generally, two kinds of effects are required to be modeled: i) Static - due to the relative wind velocity and the orientation of the airplane with respect to the wind, and ii) Dynamic - due to the relative angular motion between the airplane and the wind. It is important to note that the aerodynamic forces do not depend on the orientation of the airplane with respect to the Earth (inertial axis) as given by the angles such as θ. *Equally, they do not depend on the airplane angular rates with respect to the Earth, such as q.*

Therefore, the perturbed pitching moment ΔC_m in the most general case is modeled as a function of:

- the perturbation in Mach number ΔMa, which in case of pitch motion may be dropped as the velocity is being considered to be constant, and the perturbation in angle of attack, $\Delta\alpha$ - *static effect*
- the difference between the perturbations in body-axis and wind-axis pitch rates, $(\Delta q_b - \Delta q_w)$ - *dynamic effect*
- the perturbation in wind-axis pitch rate, Δq_w - *flow curvature effect*. To understand this, imagine an airplane performing a vertical loop of radius R with fixed velocity V^*. Then, $\Delta q_w = V^*/R$. Any component, such as the horizontal tail, located at a height h above the airplane's center-line which contains its cg, must therefore travel along the curved trajectory at a lower velocity given by $\Delta q_w(R-h) = V^* - \Delta q_w h$. The aerodynamic forces on that component will then be a function of $(V^* - \Delta q_w h)^2$, and hence dependent on Δq_w.
- *the downwash lag effect*, which is modeled as a function of the angle of attack rate $\Delta\dot{\alpha}$

Formally, one may write :

$$\Delta C_m = \Delta C_m(\Delta\alpha, \Delta q_b - \Delta q_w, \Delta q_w, \Delta\dot{\alpha}) \tag{8a}$$

The simplest and most obvious form of this function, having assumed small perturbations, is obviously a linear expansion in Taylor series as:

$$\Delta C_m = C_{m\alpha}\Delta\alpha + C_{mq1}(\Delta q_b - \Delta q_w)(c/2V)$$

$$+ C_{mq2}\Delta q_w(c/2V) + C_{m\dot{\alpha}}\Delta\dot{\alpha}(c/2V) \tag{8b}$$

where the coefficients to the terms on the right-hand side are partial derivatives (called aerodynamic derivatives or stability derivatives) defined at the chosen trim state (indicated by the '*'):

$$C_{m\alpha} = \frac{\partial C_m}{\partial\alpha}\Big|_*, \quad C_{mq1} = \frac{\partial C_m}{\partial\left[(q_b - q_w)(c/2V)\right]}\Big|_*,$$

$$C_{mq2} = \frac{\partial C_m}{\partial q_w(c/2V)}\Big|_*, \quad C_{m\dot{\alpha}} = \frac{\partial C_m}{\partial[\dot{\alpha}(c/2V)]}\Big|_* \tag{8c}$$

As long as motion is confined to the longitudinal plane, it can be easily established that

$$\Delta q_b = \Delta\dot{\theta}, \Delta q_w = \Delta\dot{\gamma}, \Delta q_b - \Delta q_w = \Delta\dot{\theta} - \Delta\dot{\gamma} = \Delta\dot{\alpha} \tag{9}$$

Hence, the aerodynamic model in Eq.(8b) can be updated as

$$\Delta C_m = C_{m\alpha}\Delta\alpha + C_{mq1}\Delta\dot{\alpha}(c/2V)$$

$$+ C_{mq2}\Delta\dot{\gamma}(c/2V) + C_{m\dot{\alpha}}\Delta\dot{\alpha}(c/2V) \tag{10a}$$

The derivative C_{mq2} is usually negligible for most airplanes except for a few special cases such as those with T-tails. We may therefore justifiably drop this term from the aerodynamic model of Eq.(10a) to arrive at a usable model as:

$$\Delta C_m = C_{m\alpha}\, \Delta\alpha + (C_{mq1} + C_{m\dot{\alpha}})\, \Delta\dot{\alpha}\, (c/2V) \qquad (10b)$$

A Note

At this point it may be instructive to pause and compare the form of the aerodynamic model in Eq.(10) with the traditional version which is modeled as

$$\Delta C_m = C_{m\alpha}\, \Delta\alpha + C_{mq1}\, \Delta q_b\, (c/2V) + C_{m\dot{\alpha}}\, \Delta\dot{\alpha}\, (c/2V)\ ,$$

where for the sake of consistency of notation, the usual derivative C_{mq} has been written as C_{mq1}. Using the relations in Eq.(9), one may render this as follows:

$$\Delta C_m = C_{m\alpha}\, \Delta\alpha + (C_{mq1} + C_{m\dot{\alpha}})\, \Delta\dot{\alpha}\, (c/2V) + C_{mq1}\, \Delta\dot{\gamma}\, (c/2V)\ .$$

The comparison clearly reveals that the final term involving flow curvature is incorrectly modeled in the traditional version - unwittingly the same physics that causes C_{mq1} has been applied to the flow curvature term, falsely creating a much stronger flow curvature effect than exists in reality. This has happened because the second term has traditionally been incorrectly modeled as $C_{mq1}\, \Delta q_b\, (c/2V)$ instead of what should be $C_{mq1}\, (\Delta q_b - \Delta q_w)\, (c/2V)$ .This error has also been traditionally carried over to the simulation of the complete 6-degree of freedom equations of airplane motion, such as the ones used in flight testing, pilot training, and control law design. Thus, the implications of this correction extend beyond the small-perturbation theory presented in the classroom.

Short Period Mode Dynamics

Inserting the perturbed pitching moment coefficient model of Eq. (10b) into the pitch dynamics Eq.(7c) gives us the short period dynamics as:

$$\Delta\ddot{\alpha} - \left\{\left(\frac{\bar{q}Sc}{I_{yy}}\right)(c/2V)\,(C_{mq1} + C_{m\dot{\alpha}})\right\}\Delta\dot{\alpha} - \left\{\left(\frac{\bar{q}Sc}{I_{yy}}\right)C_{m\alpha}\right\}\Delta\alpha = 0 \qquad (11a)$$

which is a second-order linear equation in the perturbed angle of attack, $\Delta\alpha$. From this, one can immediately obtain expressions for the frequency and damping of the short period mode as:

$$(\omega_n^2)_{sp} = -\left(\frac{\bar{q}Sc}{I_{yy}}\right)C_{m\alpha} \qquad (11b)$$

$$(2\,\zeta\omega_n)_{sp} = -\left(\frac{\bar{q}Sc}{I_{yy}}\right)(c/2V)\,(C_{mq1} - C_{m\dot{\alpha}}) \qquad (11c)$$

Note that the short period frequency in Eq.(11b) is a function of $C_{m\alpha}$ alone unlike the traditional solution where an additional erroneous term involving C_{mq} and $C_{L\alpha}$ appears. The sign of that additional term is such that even when $C_{m\alpha}$ is marginally positive (supposedly a case of static instability, or negative pitch stiffness), the combination of terms can traditionally yield a positive ω_n^2, hence a positive pitch stiffness. In contrast, the correct version in Eq.(11) ensures that $C_{m\alpha} = 0$ corresponds to zero pitch stiffness - in other words, an eigenvalue (pole) at the origin of the complex plane, as can be verified from Eq.(13) below.

Short Period Mode Dynamics with Elevator Control

It is a simple matter to extend the model in Eq.(10) to include the effect of small-perturbation elevator deflection $\Delta\delta e$ about the trim elevator state δe^*. The modified Eq.(10b) appears as:

$$\Delta C_m = C_{m\alpha}\, \Delta\alpha + (C_{mq1} + C_{m})\, \Delta\dot{\alpha}\, (c/2V) + C_{m\delta e}\, \Delta\delta e \qquad (12a)$$

where the partial derivative $C_{m\delta e}$, called the elevator or pitch control derivative, is defined as

$$C_{m\delta e} = \frac{\partial C_m}{\partial \delta e}\Big|_* \qquad (12b)$$

and the resulting model for the "forced" short period dynamics is written as:

$$\Delta\ddot{\alpha} - \left(\frac{\bar{q}Sc}{I_{yy}}\right)(c/2V)\,(C_{mq1} + C_{m\dot{\alpha}})\,\Delta\dot{\alpha} - \left(\frac{\bar{q}Sc}{I_{yy}}\right)C_{m\alpha}\,\Delta\alpha = \left(\frac{\bar{q}Sc}{I_{yy}}\right)C_{m\delta e}\,\Delta\delta e \qquad (12c)$$

Starting from this point, one may examine the effect of small elevator deflections on the pitch dynamics, either by numerical simulation of Eq.(12c), or by transforming Eq.(12c) into the Laplace domain to write the transfer function between perturbed angle of attack and perturbed elevator deflection as:

$$\frac{\Delta\alpha(s)}{\Delta\delta e(s)} = \frac{\left(\frac{\bar{q}Sc}{I_{yy}}\right) C_{m\delta e}}{s^2 - \left(\frac{\bar{q}Sc}{I_{yy}}\right)(c/2V)\,(C_{mq1} + C_{m\dot{\alpha}})\, s - \left(\frac{\bar{q}Sc}{I_{yy}}\right) C_{m\alpha}} \tag{13}$$

Then standard tools of frequency domain analysis may be used to study Eq.(13). This track is more likely to be followed in a course with focus on flight control. On the other hand, a course geared towards aerodynamic design of airplanes would focus on the aerodynamic derivatives in Eq.(12a), and that is what we consider next.

Aerodynamic Coefficients for a Wing-Body-Tail Configuration

It is standard to consider an airplane configuration consisting of wing, body (fuselage) and tail, and write out analytical expressions assuming linear aerodynamics for the lift and pitching moment coefficients. These generally appear as below. For the lift coefficient:

$$C_L = C_{L0} + C_{L\alpha}\alpha + C_{L\dot{\alpha}}\,\dot{\alpha}\,(c/2V^*) + C_{L\delta e}\,\delta e + C_{Lq1}\,(q_b - q_w)\,(c/2V) \tag{14a}$$

where

$$C_{L0} = C^{wb}_{L\alpha}\,\alpha_0 - \frac{S_t}{S}\,C^t_{L\alpha}\,(i_t + \varepsilon_0) \tag{14b}$$

$$C_{L\alpha} = C^{wb}_{L\alpha} + \frac{S_t}{S}\,C^t_{L\alpha}\,(1 - \varepsilon_\alpha) \tag{14c}$$

$$C_{L\dot{\alpha}} = -\frac{S_t}{S}\,C^t_{L\alpha}\,(c/2v^8)\,\varepsilon_{\dot{\alpha}} \tag{14d}$$

$$C_{L\delta e} = \left(\frac{S_t}{S}\right) C^t_{L\delta e} \tag{14e}$$

$$C_{Lq1} = 2\,V_H\,C^t_{L\alpha} \tag{14f}$$

where the C_L derivatives are defined in a manner similar to those for C_m in Eqs.(8c) and (12b). For the pitching moment coefficient :

$$C_{mCG} = C_{m0} + C_{m\alpha}\alpha + C_{m\dot{\alpha}}\,\dot{\alpha}\,(c/2V^*) + C_{m\delta e}\,\delta e + C_{mq1}\,(q_b - q_w)\,(c/2V^*) \tag{15a}$$

where

$$C_{m0} = C^{wb}_{mAC} + (h_{CG} - h^{wb}_{AC})\,C_{L0} + V'_H\,C^t_{L\alpha}\,(i_t + \varepsilon_0) \tag{15b}$$

$$C_{m\alpha} = (h_{CG} - h^{wb}_{AC})\,C_{L\alpha} - V'_H\,C^t_{L\alpha}\,(1 - \varepsilon_\alpha) \tag{15c}$$

$$C_{m\dot{\alpha}} = (h_{CG} - h^{wb}_{AC})\,C_{L\dot{\alpha}} + V'_H\,C^t_{L\alpha}\,(c/2V^*) \tag{15d}$$

$$C_{m\delta e} = (h_{CG} - h^{wb}_{AC})\,C_{L\delta e} - V'_H\,C^t_{L\delta e} \tag{15e}$$

$$C_{mq1} = (h_{CG} - h^{wb}_{AC})\,C_{Lq1} - 2V'_H\,C^t_{L\alpha}\,\left(\frac{l_t}{c}\right) \tag{15f}$$

The downwash angle at the horizontal tail, ε_{tail}, has been modeled by a linear approximation:

$$\varepsilon_{tail}(t) = \varepsilon_0 + \varepsilon_\alpha\,\alpha + \varepsilon_{\dot{\alpha}}\,\dot{\alpha}\,(c/2V^*) \tag{16a}$$

Some of the parameters of interest are the ratio of tail to wing planform area S_t/S, tail setting angle i_t, non-dimensional cg location h_{CG}, non-dimensional wing-body aerodynamic center location h^{wb}_{AC}, and the horizontal tail volume ratio (HTVR) defined as:

$$V'_H = \frac{S_t\,l'_t}{Sc}, \quad V_H = V'_H - \frac{S_t}{S}\,(h_{CG} - h^{wb}_{AC}),$$

$$\frac{l_t}{c} = \frac{l'_t}{c} - (h_{CG} - h^{wb}_{AC}) \tag{16b}$$

where l'_t is the distance between the tail aerodynamic center and the wing-body aerodynamic center, and l_t is the distance from the tail aerodynamic center to the airplane cg. Note that l'_t is relatively fixed whereas l_t varies with shift in airplane cg.

Equations (14) and (15) may be used to discuss airplane trim, as given by Eq.(6), and the stability of the short period mode dynamics, as may be obtained from Eq.(11). The requirements of positive pitch stiffness and positive pitch damping for short period stability translates into:

$$C_{m\alpha} < 0 \quad \text{and} \quad (C_{mq1} + C_{m\dot{\alpha}}) < 0 \tag{17}$$

Susequently, one can discuss tail sizing and examine trim and stability in pitch under shift in cg location, which leads to the concept of the neutral point, defined as:

$$h_{NP} = h_{AC}^{wb} + V_H' \, (C_{L\alpha}' / C_{L\alpha}) \, (1 - \varepsilon_\alpha) \tag{18}$$

The expressions in Eqs.(15b) and (15c) may then be rewritten in terms of h_{NP} as follows :

$$C_{m0} = C_{mNP} + C_{L0} \, (h_{CG} - h_{NP}) \tag{19a}$$

$$C_{m\alpha} = C_{L\alpha} \, (h_{CG} - h_{NP}) \tag{19b}$$

whereby the static part of C_{mCG} in Eq.(15a) may be transformed to appear as :

$$C_{mCG} = C_{mNP} + C_L \, (h_{CG} - h_{NP}) \tag{19c}$$

An important point here is that this entire discussion of trim and stability naturally relates to the short period dynamics, with the stability criterion in Eq.(17) coming directly from Eq.(11) for the short period mode. Further, when the cg is at the neutral point, Eq.(19b) clearly shows $C_{m\alpha}$=0 which implies zero pitch stiffness from Eq.(11).

Following this, one can talk about the use of elevator to trim at different V^* or α^*, and derive a relation between the trim lift coefficient C_L^* and the trim angle of attack α^*, and point out how this relation differs from the purely aerodynamic model in Eq.(14a) which has no reference to the pitching moment balance in Eq.(6c) at a trim state. This leads to a determination of the forwardmost allowable cg position for an airplane which usually corresponds to the maximum 'up' deflection of the elevator. Several other topics of interest can follow from this point:

- An alternative notion of stability based on the change in trim angle of attack per change in trim elevator deflection.
- The determination of neutral point from flight tests.
- Effect of neutral point shift with Mach number on airplane trim and stability.

For a complete discussion of these matters, please refer to the textbook (Sinha and Ananthkrishnan [20]).

Aerodynamic Coefficients for an Airplane in Real Life

In practice the lift and moment coefficients for an airplane are obtained by wind tunnel tests and it is worthwhile to present an example. Figs.4 and 5 show the variation of C_L and C_m with angle of attack for a military airplane over a range of α from -14 to +90 deg. The expressions in Eqs.(14) and (15) are linear approximations at low angles of attack, marked by the first dashed line in Figs.4 and 5. Beyond that, nonlinear effects come in and a different linear approximation may be used up to stall, as marked by the second dashed line in Figs.4 and 5 - in the present example, the lift curve slope decreases and the pitching moment slope becomes less negative. Stall occurs at around 35 deg angle of attack beyond which lift falls gradually but a steep increase in negative pitching moment is observed.

Heave/ Phugoid Dynamics

The heave dynamics occurs at the slower timescale T_2 given by Eq.(5b) and can hence be studied independently of the faster pitch dynamics. To study the heave dynamics, we take Eqs.(4a) and (4b) and consider small perturbations in velocity ΔV and in flight path angle $\Delta\gamma$, while holding the pitch variable, angle of attack, fixed at its trim value α^*. Thus, θ^*-γ^*= α^*, and $\Delta\theta$=$\Delta\gamma$. Perturbations in thrust are ignored, which is generally acceptable for jet-powered airplanes.

Writing out the small-perturbation equations for the heaving motion, we have:

$$\dot{V}^* + \Delta V = g \left\{ \frac{T}{W} \cos \alpha^* - \frac{\bar{q}^* S}{W} \left(1 + 2 \frac{\Delta V}{V^*} \right) (C_D^* + \Delta C_D) - \sin (\gamma^* + \Delta\gamma) \right\} \tag{20a}$$

$$\Delta \dot{\gamma} (\dot{V}^* + \Delta V) = g \left\{ \frac{T}{W} \sin \alpha^* + \frac{\bar{q}^* S}{W} \left(1 + 2 \frac{\Delta V}{V^*} \right) (C_L^* + \Delta C_L) - \cos (\gamma^* + \Delta\gamma) \right\} \tag{20b}$$

where we have used the following relation for the perturbation in the dynamic pressure $\bar{q}$:

$$\bar{q} = \bar{q}^* \, (1 + 2 \frac{\Delta V}{V^*}) \tag{20c}$$

Eliminating the trim condition in Eqs.(6a) and (6b) from Eq.(20) yields the final form of the heave perturbation equations as:

$$\frac{\Delta \dot{V}}{V^*} = \left(\frac{g}{V^*}\right)\left\{ - \frac{\bar{q}^* S}{W}\left(\Delta C_D + 2C_D^* \frac{\Delta V}{V^*}\right) - \cos\gamma^* \Delta\gamma\right\} \tag{21a}$$

$$\Delta\dot{\gamma} = \left(\frac{g}{V^*}\right)\left\{\frac{\bar{q}^* S}{W}\left(\Delta C_L + 2C_L^* \frac{\Delta V}{V^*}\right) - \sin\gamma^* \Delta\gamma\right\} \tag{21b}$$

Which brings us to the question of modeling the perturbed lift and drag coefficients, ΔC_L and ΔC_D.

Small-perturbation Aerodynamic Modeling (Heave Motion)

As in the case of ΔC_m, there are four possible aerodynamic effects we must consider while modeling ΔC_L and ΔC_D:

- ***Static effect***: It is known that the lift and drag coefficients are functions of the Mach number. Hence the effect of change in Mach number, ΔMa, must be included. However, since a constant angle of attack α^* has been assumed for the heave motion, there is no effect due to $\Delta\alpha$.
- ***Dynamic effect***: Due to the difference between the perturbations in body-axis and wind-axis pitch rates, (Δq_b - Δq_w), which is equal to $\Delta\dot{\alpha}$ from Eq.(9), and therefore plays no role in the heave model.
- ***Flow curvature effect***: Due to the perturbation in wind-axis pitch rate, Δq_w, which is equal to $\Delta\dot{\gamma}$ from Eq.(9), and hence must be included. However, as we have seen, this effect is not significant for many conventional airplane configurations, and can therefore be dropped except for special cases such as T-tails.
- ***Downwash lag effect***, which is modeled as a function of the angle of attack rate $\Delta\dot{\alpha}$, so it does not figure in the heave dynamics either.

Effectively, the perturbed drag and lift coefficients in the case of heaving motion can be modeled as:

$$\Delta C_D = C_{DMa}\Delta Ma = C_{DMa}\frac{\Delta V}{a^*} = \frac{V^*}{a^*} C_{DMa}\left(\frac{\Delta V}{V^*}\right) = Ma^* C_{DMa}\left(\frac{\Delta V}{V^*}\right) \tag{22a}$$

And in similar fashion,

$$\Delta C_L = C_{LMa}\Delta Ma = Ma^* C_{LMa}\left(\frac{\Delta V}{V^*}\right) \tag{22b}$$

assuming a^* to be constant despite the small change in altitude due to the heaving motion. The aerodynamic derivatives in Eq.(22) are defined as below:

$$C_{DMa} = \frac{\partial C_D}{\partial Ma}\Big|_* , \quad C_{LMa} = \frac{\partial C_L}{\partial Ma}\Big|_* \tag{23}$$

The '*' indicates that the partial derivatives are to be evaluated at the chosen trim state.

Phugoid Mode Dynamics

Replacing the perturbed aerodynamic coefficients in Eq.(21) with the model in Eq.(22), we can write:

$$\frac{\Delta \dot{V}}{V^*} = \left(\frac{g}{V^*}\right)\left\{ - \frac{\bar{q}^* S}{W}\left(Ma^* C_{DMa} + 2C_D^*\right)\frac{\Delta V}{V^*} - \cos\gamma^* \Delta\gamma\right\} \tag{24a}$$

$$\Delta\dot{\gamma} = \left(\frac{g}{V^*}\right)\left\{\frac{\bar{q}^* S}{W}\left(Ma^* C_{LMa} + 2C_L^*\right)\frac{\Delta V}{V^*} + \sin\gamma^* \Delta\gamma\right\} \tag{24b}$$

The next step is to combine Eq.(24a) and (24b) into a single equation. To do this, first differentiate Eq.(24a) with time and replace $\Delta\dot{\gamma}$ on the right-hand side by Eq.(24b). Further, assume the trim state to be one of level flight. Hence, $\cos\gamma^*=1$ and $\sin\gamma^*=0$. This yields a single second-order differential equation in ΔV for the phugoid mode dynamics as:

$$\frac{\Delta \ddot{V}}{V^*} + \left(\frac{g}{V^*}\right)\left[\frac{\bar{q}^* S}{W}\left(Ma^* C_{DMa} + 2C_D^*\right)\right]\frac{\Delta \dot{V}}{V^*} + \left(\frac{g}{V^*}\right)^2\left[\frac{\bar{q}^* S}{W}\left(Ma^* C_{LMa} + 2C_L^*\right)\right]\frac{\Delta V}{V^*} = 0 \tag{25}$$

To extract the frequency and damping of the phugoid mode from Eq.(25) is a simple task:

$$(\omega_n^2)_p = \left(\frac{g}{V^*}\right)^2 \left[\frac{\bar{q}^* S}{W} \left(Ma^* C_{LMa} + 2C_L^* \right) \right] \quad (26a)$$

$$(2\zeta\omega_n)_p = \left(\frac{g}{V^*}\right) \left[\frac{\bar{q}^* S}{W} \left(Ma^* C_{DMa} + 2C_D^* \right) \right] \quad (26b)$$

Before we examine the stability of the phugoid mode from Eq.(26), we need to expand on the aerodynamic derivatives with Mach number.

Aerodynamic Coefficients as a Function of Mach Number

The variation of the lift coefficient with Mach number is usually given by the Prandtl-Glauert rule:

$$C_L(Ma) = \frac{C_{L(Ma=0)}}{\sqrt{1 - Ma^2}} \text{ (in subsonic flow)} \quad (27a)$$

Differentiating Eq.(27a) with Mach number, one can derive,

$$C_{LMa} = \frac{\partial C_L}{\partial Ma}\Big|_* = C_L(Ma^*) \cdot \frac{Ma^*}{(1 - Ma^{*2})} > 0 \text{ (in subsonic flow)} \quad (27b)$$

Thus, every term on the right-hand side of Eq.(26a) for the phugoid frequency is ordinarily positive (at least in subsonic flight), hence heave stiffness is virtually guaranteed for most airplanes. The requirement for phugoid mode stability then boils down to the following condition on the phugoid damping:

$$\left(\frac{g}{V^*}\right) \left[\frac{\bar{q}^* S}{W} \left(Ma^* C_{DMa} + 2C_D^* \right) \right] > 0 \quad (28)$$

which is also almost certainly true in low-speed flight since $C_{DMa} \approx 0$ and the other term is positive. Therefore, phugoid mode stability under these conditions is usually not a serious issue.

At high supersonic speeds, however, C_{LMa} may be negative and the $Ma^* C_{LMa}$ term may overwhelm the $2C_L^*$ term in Eq.(26a) to give negative stiffness in heave. Also, for wave-drag-dominant configurations, $C_{DMa} < 0$, and the heave mode damping in Eq.(26b) may also turn out negative. Therefore, it is not unusual for very high-speed airplanes to have a poorly damped or even unstable phugoid mode.

At very low speeds, one may neglect $Ma^* C_{LMa}$ in comparison to $2C_L^*$, and $Ma^* C_{DMa}$ as compared to $2C_D^*$ in Eq.(26). That gives further approximate versions of the phugoid frequency and damping formulas as:

$$(\omega_n)_p \approx \sqrt{2} \left(\frac{g}{V^*}\right), (2\zeta\omega_n)_p \approx \left(\frac{g}{V^*}\right) \left[\frac{\bar{q}^* S}{W} \left(2C_D^* \right) \right]$$

$$= \left(\frac{g}{V^*}\right) \left(\frac{2C_D^*}{C_L^*} \right) \text{ or } \zeta_p \approx \frac{1}{\sqrt{2}} \frac{C_D^*}{C_L^*} \quad (29)$$

From Eq.(26) or Eq.(29), it is obvious that the phugoid frequency and damping are inversely related to the trim velocity V^*. In fact, this conclusion follows directly once the slow timescale $T_2 = V/g$ is identified with the phugoid mode dynamics. Another interesting observation is that the phugoid damping is inversely proportional to the aerodynamic efficiency at trim, C_L^*/C_D^*. Full flap deflection used at landing approach therefore has the effect of improving phugoid damping.

Whereas the short period frequency and damping in Eq.(11) usually provide a close match to values observed in flight, the phugoid formulas in Eq.(26) do not always give a quantitatively correct approximation to the in-flight frequency and damping. The mystery was resolved by Ananthkrishnan and Unnikrishnan [16] who pointed out that there is a component of $\Delta\alpha$, called the static residual, that persists even after the short period mode has died out and varies as per the phugoid timescale. When the static residual is included in the multiple timescale approach, improved approximations are obtained to the phugoid mode parameters, and indeed in similar fashion to the slower modes in the lateral-directional dynamics as well.

Improved Longitudinal Mode Approximations

The complete set of small-perturbation longitudinal equations assuming a level flight trim state may be gathered from Eqs.(21) and (7b) as below:

$$\frac{\Delta \dot{V}}{V^*} = \left(\frac{g}{V^*}\right) \left\{ - \frac{\bar{q}^* S}{W} \left(\Delta C_D + 2C_D^* \frac{\Delta V}{V^*} \right) - \Delta\gamma \right\} \quad (30a)$$

$$\Delta\dot{\gamma} = \left(\frac{g}{V^*}\right)\left\{\frac{\bar{q}^*S}{W}\left(\Delta C_L + 2C_L^*\frac{\Delta V}{V^*}\right)\right\} \quad (30b)$$

$$\Delta\ddot{\theta} = \left(\frac{\bar{q}Sc}{I_{yy}}\right)\Delta C_m \quad (30c)$$

The most general model for the perturbed aerodynamic coefficients may be written as follows:

$$\Delta C_D = C_{DMa}\Delta Ma + C_{D\alpha}\Delta\alpha + C_{Dq1}(\Delta q_b - \Delta q_w)(c/2V^*) + C_{Dq2}\Delta q_w(c/2V^*) = Ma^*C_{DMa}\left(\frac{\Delta V}{V^*}\right) + C_{D\alpha}\Delta\alpha + C_{Dq1}\Delta\dot{\alpha}(c/2V^*) + C_{Dq2}\Delta\dot{\gamma}(c/2V^*)$$

$$\Delta C_L = C_{LMa}\Delta Ma + C_{L\alpha}\Delta\alpha + C_{Lq1}(\Delta q_b - \Delta q_w)(c/2V^*) + C_{Lq2}\Delta q_w(c/2V^*) = Ma^*C_{LMa}\left(\frac{\Delta V}{V^*}\right) + C_{L\alpha}\Delta\alpha + C_{Lq1}\Delta\dot{\alpha}(c/2V^*) + C_{Lq2}\Delta\dot{\gamma}(c/2V^*)$$

$$\Delta C_m = C_{mMa}\Delta Ma + C_{m\alpha}\Delta\alpha + C_{mq1}(\Delta q_b - \Delta q_w)(c/2V^*) + C_{mq2}\Delta q_w(c/2V^*) = Ma^*C_{mMa}\left(\frac{\Delta V}{V^*}\right) + C_{m\alpha}\Delta\alpha + C_{mq1}\Delta\dot{\alpha}(c/2V^*) + C_{mq2}\Delta\dot{\gamma}(c/2V^*) \quad (31)$$

where the elevator control derivatives have not been included for convenience.

Short Period Mode Dynamics

First we study the fast pitch dynamics at timescale T_1 in Eq.(30c). In this interim, we can assume the slower variables - ΔV, $\Delta\gamma$ - to be effectively constant, hence their time rates of change are zero. That allows us to write the perturbed pitch dynamics as:

$$\Delta\ddot{\alpha} = \left(\frac{\bar{q}Sc}{I_{yy}}\right)\Delta C_m = \left(\frac{\bar{q}Sc}{I_{yy}}\right)\left\{Ma^*C_{mMa}\left(\frac{\Delta V}{V^*}\right) + C_{m\alpha}\Delta\alpha + C_{m\dot{\alpha}\dot{\alpha}}\Delta\dot{\alpha}(c/2V^*)\right. \quad (32a)$$

Re-arranging terms, we obtain the short period dynamic model as :

$$\Delta\ddot{\alpha} - \left(\frac{\bar{q}Sc}{I_{yy}}\right)(c/2V)(C_{mq1} + C_{m\dot{\alpha}})\Delta\dot{\alpha} - \left(\frac{\bar{q}Sc}{I_{yy}}\right)C_{m\alpha}\Delta\alpha = \left(\frac{\bar{q}Sc}{I_{yy}}\right)Ma^*C_{mMa}\left(\frac{\Delta V}{V^*}\right) \quad (32b)$$

The left-hand side of Eq.(32b) is identical to Eq.(11a) and hence the short period frequency and damping expressions in Eq.(11) are retained unchanged. However, the presence of the slow term on the right-hand side of Eq.(32c) implies that the perturbation in angle of attack does not die down to zero; instead it leaves behind a static residual value:

$$\Delta\alpha_s = -Ma^*(C_{mMa}/C_{m\alpha})\left(\frac{\Delta V}{V^*}\right) \quad (33)$$

which now varies at the slower timescale T_2.

Phugoid Mode Dynamics

Combining Eqs.(30a) and (30b) at the slower timescale with the aerodynamic model in Eq.(31) yields a single second-order equation for the phugoid dynamics, as in Eq.(25), as follows:

$$\frac{\Delta\ddot{V}}{V^*} = \left(\frac{g}{V^*}\right)\left[\frac{\bar{q}^*S}{W}\left(Ma^*C_{DMa} + 2C_D^*\right)\right]\frac{\Delta\dot{V}}{V^*} + \left(\frac{g}{V^*}\right)\frac{\bar{q}^*S}{W}C_{D\alpha}\Delta\dot{\alpha} + \left(\frac{g}{V^*}\right)^2\left[\frac{\bar{q}^*S}{W}\left(Ma^*C_{LMa} + 2C_L^*\right)\right]\frac{\Delta V}{V^*} + \left(\frac{g}{V^*}\right)^2\frac{\bar{q}^*S}{W}C_{L\alpha}\Delta\alpha = 0 \quad (34a)$$

where we drop the flow curvature terms as before, as well as the $q1$ and $\dot{\alpha}$ derivatives - C_{Dq1} is usually not so important and C_{Lq1} gives a correction to the phugoid damping that is of higher order in g/V^* and hence may be ignored. The Δa in Eq.(34a) is the component that varies at the slow timescale and which we have determined as the static residual in Eq.(33). $\Delta\dot{\alpha}$ may be obtained by differentiating Eq.(33). The resulting model for the phugoid mode dynamics appears as:

$$\frac{\Delta\ddot{V}}{V^*} = \left(\frac{g}{V^*}\right)\left[\frac{\bar{q}^*S}{W}\left(Ma^*C_{DMa} + 2C_D^* - Ma^*(C_{mMa}/C_{m\alpha})C_{D\alpha}\right)\right]\frac{\Delta\dot{V}}{V^*} + \left(\frac{g}{V^*}\right)^2\left[\frac{\bar{q}^*S}{W}\left(Ma^*C_{LMa} + 2C_L^* - Ma^*(C_{mMa}/C_{m\alpha})C_{L\alpha}\right)\right]\frac{\Delta V}{V^*} = 0 \quad (34b)$$

The expressions for the phugoid frequency and damping may be read off from Eq.(34b) as follows:

$$(\omega_n^2)_p = \left(\frac{g}{V^*}\right)^2 \left[\frac{\bar{q}^* S}{W}\left(Ma^* C_{LMa} + 2C_L^* - Ma^* (C_{mMa}/C_{m\alpha})\, C_{L\alpha}\right)\right] \tag{34c}$$

$$(2\zeta\omega_n)_p = \left(\frac{g}{V^*}\right)\left[\frac{\bar{q}^* S}{W}\left(Ma^* C_{DMa} + 2C_D^* - Ma^* (C_{mMa}/C_{m\alpha})\, C_{D\alpha}\right)\right] \tag{34d}$$

And these are different from the previous formulas in Eq.(26) by one additional term each which may be quite significant. C_{La} is a fairly large number, usually of the order of 5 to 6 /rad. C_D can be expressed as a generic drag polar as follows:

$$C_D = C_{D0} + K_1 C_L + K_2 C_L^2 \tag{35a}$$

Which on differentiating gives,

$$C_{Da} = (K_1 + 2K_2 C_L^*)\, C_{La} \tag{35b}$$

C_{nMa} mainly arises due to the change in lift coefficient at the horizontal tail by the Prandtl-Glauert rule as in Eq.(27a). Another reason for change of C_m with Ma is due to the shift in the aerodynamic center (and hence the neutral point) as the airplane traverses the transonic flight regime. The np shifts further aft in supersonic flight; hence the net lift acting at the airplane np exerts a larger down-pitching moment at the airplane cg. As the airplane traverses up the transonic Mach numbers, this down-pitching moment gradually comes into effect pitching the airplane nose down. This is often called the *tuck under effect*.

Effectively, the inclusion of the static residual is crucial in getting improved approximations to the phugoid mode parameters.

Conclusion

This paper has suggested improvements and corrections to the existing presentation of airplane flight dynamics from a pedagogical perspective. Most promimently, the modeling of the dynamic (rate) derivatives, a carry over from the original work by Bryan [1], has been shown to be faulty and has been replaced by a new model that bears greater fidelity to aerodynamic theory. It is argued that it is better to start with the easily-derived equations of motion in the longitudinal plane than the complete six degree of freedom equations which puts off many in the audience. The fast and slow timescales in longitudinal dynamics are clearly identified and are used to derive the equations for the short period and phugoid modes - once without considering the static residual, which gives the standard, poor phugoid approximations, and then again with the static residual, which gives improved expressions for the phugoid frequency and damping. The redundant and confusing concepts of dimensional derivatives and static stability have been junked. It is hoped that this simple, corrected presentation of flight dynamics will gain pedagogical acceptance.

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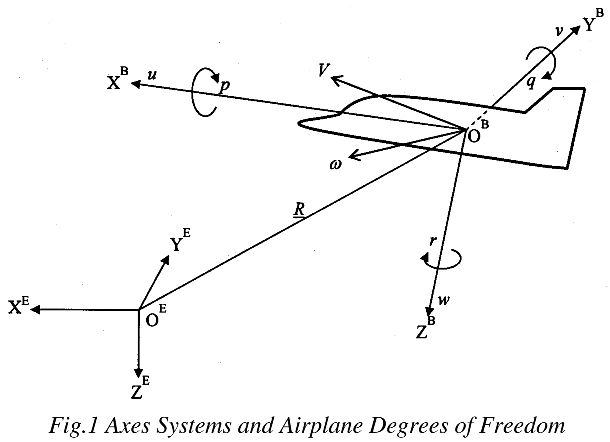

Fig.1 Axes Systems and Airplane Degrees of Freedom Represented by the Vectors V and ω

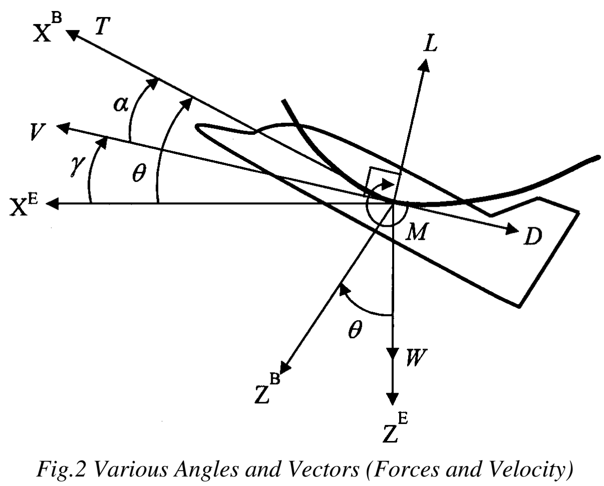

Fig.2 Various Angles and Vectors (Forces and Velocity) in the Longitudinal Plane

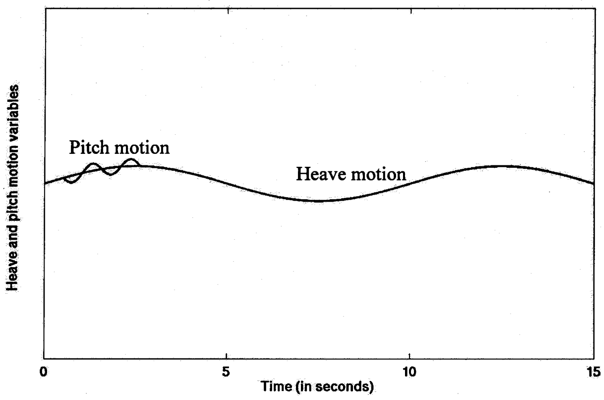

Fig.3 Heave Motion Variable Changes Little During the Short Time Over which the Pitch Motion Variable is Excited and Quickly Damps Out

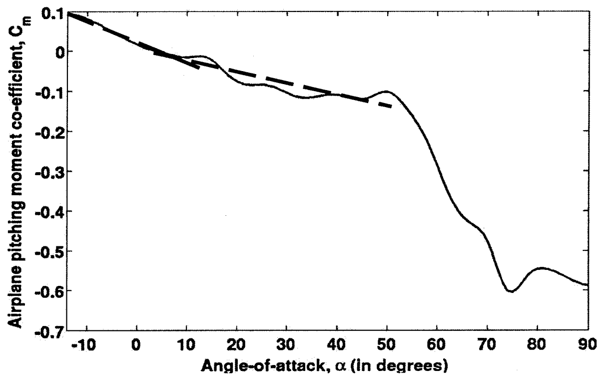

Fig.5 Airplane Pitching Moment Coefficient as a Function of Angle of Attack

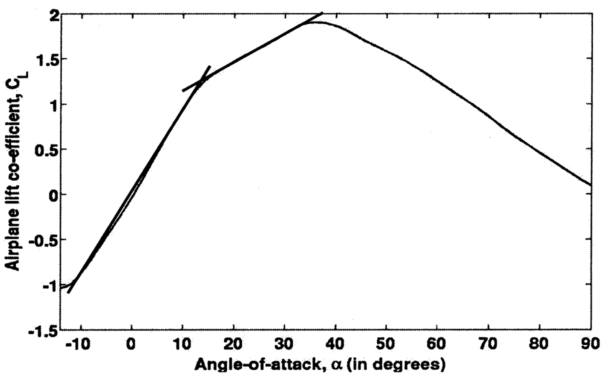

Fig.4 Example Airplane Lift Coefficient as a Function of Angle of Attack